HOMOTOPY TREES WITH TRIVIAL CLASSIFYING RING

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ABSTRACT. In this note we study a certain class of groups π for which the homotopy classification of (π, m) -complexes is independent of the k -invariant for small m .

1. Homotopy trees. A (π, m) -complex is a finite, connected m -dimensional CW-complex such that $\pi_1(X) = \pi$ and $\pi_i(X) = 0$ for $1 < i < m$. The homotopy tree $HT(\pi, m)$ is a directed tree whose vertices are homotopy classes of (π, m) -complexes. A vertex $[X]$ is connected by an edge to a vertex $[Y]$ iff $Y \cong X \vee S^m$. The problem is to describe $HT(\pi, m)$ (see [3]).

There is an algebraic analog to this problem in the theory of algebraic m -types. The algebraic m -type $\Pi(X)$ of a (π, m) -complex X is a triple $\Pi(X) = (\pi, \pi_m(X), k(X))$ where $k(X) \in H^{m+1}(\pi, \pi_m(X))$ is the first k -invariant of X (see [7], [3]). It is proved in [4], that if $\pi_m = \pi_m(X)$ is finitely generated as a π -module and $H^{m+1}(\pi; Z\pi) = 0$, then $R(\pi, m) = H^{m+1}(\pi, \pi_m)$ has the structure of a ring with identity such that the units $U(\pi, m)$ of $H^{m+1}(\pi, \pi_m)$ are the projective k -invariants. Note that $H^{m+1}(\pi, \pi_m(X)) \cong H^{m+1}(\pi, \pi_m(Y))$ for any two (π, m) -complexes X and Y , by Schanuel's lemma. In §2 we isolate a large class of groups π having $H^i(\pi; Z\pi) = 0$ for all but one value of i .

Furthermore, there is a homomorphism $\kappa: U(\pi, m) \rightarrow \tilde{K}_0 Z\pi$ where $\tilde{K}_0 Z\pi$ is the reduced projective class group of the integral group ring $Z\pi$ of π . The kernel of κ , denoted $SF(\pi, m)$, is precisely the set of units which arise as the k -invariants of (π, m) -complexes.

DEFINITION. The ring $R(\pi, m)$ is called the *classifying ring* of the homotopy tree $HT(\pi, m)$; $\kappa: U \rightarrow \tilde{K}_0 Z\pi$, the *classifying homomorphism*.

We say that the classifying ring of $HT(\pi, m)$ is *trivial* if it is isomorphic to the zero ring or the ring of integers Z . In the latter case $\kappa: U \rightarrow \tilde{K}_0 Z\pi$ is the zero homomorphism.

In order to state the results, we need to define an isomorphism of algebraic m -types. $(\theta, \beta): \Pi(X) \rightarrow \Pi(Y)$ is an isomorphism if $\theta: \pi \rightarrow \pi$ is a group automorphism, $\beta: \pi_m(X) \rightarrow \pi_m(Y)$ is a θ -isomorphism of π -modules ($\beta(x \cdot y) = \theta(x)\beta(y)$, for $x \in \pi, y \in \pi_m(X)$ and β is bijective) and $\theta^{*-1}\beta_*(k(X)) = k(Y)$ in the following diagram:

$$H^{m+1}(\pi, \pi_m(X)) \xrightarrow{\beta^*} H^{m+1}(\pi, \pi_m(Y)_\theta) \xleftarrow{\theta^*} H^{m+1}(\pi, \pi_m(Y))$$

where $\pi_m(Y)_\theta$ is the π -module with action $x * y = \theta(x) \cdot y$ ($x \in \pi, y$

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$\in \pi_m(Y)$). It is known from [7] that X and Y have the same homotopy type iff $\Pi(X) \cong \Pi(Y)$.

We say that the homotopy tree $HT(\pi, m)$ is *independent of the k -invariant* if any two (π, m) -complexes X and Y having θ -isomorphic ($\theta \in \text{Aut } \pi$) m -homotopy modules have the same homotopy type.

1.1. THEOREM. *If the classifying ring $R(\pi, m)$ is trivial, then the homotopy tree $HT(\pi, m)$ is independent of the k -invariant.*

PROOF. For the proof we use the following result of S. Eilenberg and S. Mac Lane: for any (π, m) -complexes X, Y such that $(\theta, \beta): \Pi(X) \cong \Pi(Y)$ there exists a map $f: X \rightarrow Y$ realizing (θ, β) in the sense that $f_{1\#} = \theta$ and $f_{m\#} = \beta$.

The theorem is clear if $R(\pi, m) = 0$. Now suppose that $H^{m+1}(\pi, \pi_m(X)) \cong Z$ as a ring. Now $\beta_\star^{-1} \circ \theta^\star(1) = \pm 1$ because $\beta_\star^{-1} \circ \theta^\star$ is an isomorphism $Z \rightarrow Z$. Thus, as $k(X)$ and $k(Y)$ must be units in Z , one of

$$(\theta, \pm\beta): (\pi, \pi_m(X), k(X)) \rightarrow (\pi, \pi_m(Y), \pm \theta^{\star-1} \circ \beta_\star(k(X)))$$

is an isomorphism $\Pi(X) \rightarrow \Pi(Y)$. This completes (1.1).

We now consider a class of groups $\mathcal{P}^n$ ($1 < n \leq \infty$). A group G has property $\mathcal{P}^n$ if $H^i(G; ZG) = 0$ for $0 < i < n$. This is a rather large class. For example, any finite group has $\mathcal{P}^\infty$; finitely generated abelian groups of rank n have $\mathcal{P}^n$ ($n \geq 1$) [2, Proposition 3.1]. Also, duality groups of cohomological dimension n have $\mathcal{P}^n$ [1], [2], [6]. In §2, we study a restricted class $\mathcal{C}$ which is closed under extensions. From this we see that infinite polycyclic groups having Hirsch number n are $\mathcal{P}^n$ -groups.

1.2. THEOREM. *Let n be an integer ≥ 4 . Let π be an infinite group satisfying $\mathcal{P}^n$ and let $m+1 < n$. Then the classifying ring $R(\pi, m)$ is infinite cyclic.*

Let $\xi(X, Y)$ denote the set of homotopy classes of homotopy equivalences $X \rightarrow Y$; $\xi(X) \equiv \xi(X, X)$ the self-equivalence group of X .

1.3. COROLLARY. *Under the above hypotheses, let X, Y be two (π, m) -complexes having the same homotopy type. Then*

- (i) $\xi(X, Y) \cong \text{Iso}(\Pi(X), \Pi(Y))$ (set bijection),
- (ii) $\xi(X) \cong \text{Aut}(\Pi(X))$ (group isomorphism). The isomorphisms are given by carrying $(f: X \rightarrow Y) \mapsto (f_\#: \Pi(X) \rightarrow \Pi(Y))$.

A. J. Sieradski has proved the corollary for π a finite group [3, Theorem 6.6].

PROOF. Let X be any (π, m) -complex. $H^i(\pi; Z\pi) = 0$ for $0 < i \leq m+1$ implies that $H^{m+1}(\pi, \pi_m(X)) \cong H^0(\pi, Z) \cong Z$ as a ring. Also $H^m(\pi, \pi_m(X)) \cong H^0(\pi; A(\pi)) = A(\pi)^\pi = 0$ because π is infinite. Here $A(\pi)$ is the augmentation ideal. The corollary follows from Theorem 6.4 of [3] because $H^m(\pi, \pi_m(X))$ measures the number of homotopy classes of homotopy equivalences $f: X \rightarrow Y$ inducing the same isomorphism $f_\#: \Pi(X) \rightarrow \Pi(Y)$.

Two π -modules M, M' are θ -isomorphic ($\theta \in \text{Aut } \pi$) if there exists a θ -isomorphism $\beta: M \rightarrow M'$. We say that M and M' are $(\text{Aut } \pi)$ -isomorphic iff M, M' are θ -isomorphic for some $\theta \in \text{Aut } \pi$; they are *stably isomorphic* if $M \oplus (Z\pi)^i \cong M' \oplus (Z\pi)^i$ for some $i \geq 0$. Here isomorphism (" $\cong$ ") means id-isomorphism. Let $*_M$ denote the class of π -modules M' which are $(\text{Aut } \pi)$ -

isomorphic to M . A module M' is a *noncancellation example* for M iff (a) M and M' are stably isomorphic and (b) M, M' are not $(\text{Aut } \pi)$ -isomorphic. Let $\mathcal{NC}(M)$ denote the set of $(\text{Aut } \pi)$ -isomorphism classes of noncancellation examples of M together with $*_M$. Thus π -modules M^1 and M^2 belong to the same class in $\mathcal{NC}(M)$ iff M and M^i are stably isomorphic ($i = 1, 2$) and M^1, M^2 are $(\text{Aut } \pi)$ -isomorphic.

It happens in some cases that $\mathcal{NC}(M)$ is a group [3]. For example, if π is finite and M is $Z \oplus Z\pi$ (Z has the trivial module structure) then $\mathcal{NC}(Z \oplus Z\pi)$ is isomorphic to a quotient group of a subgroup of $\text{Pic}(Z\pi)$.

1.4. COROLLARY. Let $\vec{\chi}: HT(\pi, m) \rightarrow Z$ be given by $\vec{\chi}[X] = (-1)^m \chi(X)$, where $\chi(X)$ is the Euler character of the (π, m) -complex X . Under the hypotheses of 1.2, the set of homotopy types of (π, m) -complexes X having the same directed Euler character $\chi[X] = i$ is given by $\vec{\chi}^{-1}(i) = \mathcal{NC}(\pi_m(X))$.

2. $\mathcal{C}^n$ -groups.

DEFINITION. A group π has property $\mathcal{C}^n$ ($n \geq 0$) iff:

- (a) For each free π -module F , $H^i(\pi; F) = 0$ for $i \neq n$.
- (b) For each group G having π as a normal subgroup, let F' be an arbitrary free G -module. Then $H^n(\pi, F') \cong Z \otimes_{Z\pi} F'$ as a G/π -module, where Z is the trivial G -module.

2.1. PROPOSITION. (i) π is a $\mathcal{C}^0$ -group iff π is finite.

(ii) Any Poincaré duality group (PD-group) having cohomological dimension n is a $\mathcal{C}^n$ -group.

PROOF. (i) If π is an infinite $\mathcal{C}^0$ -group, then by (b), $H^0(\pi; Z\pi) \cong Z$ which is impossible. For π finite, it is well known that (a) is true because $Z\pi$ is weakly injective. To see (b), let $\pi \triangleleft G$ and F' be a free G -module. Then $H^0(\pi, F') \cong (F')^\pi \cong Z \otimes_{Z\pi} F'$ as G/π -modules.

(ii) For π a Poincaré duality group of cohomological dimension n , (a) follows from Theorem 2.3 of [2] and (b) from Lemma 2.3 of [1].

Now let $\mathcal{C}$ be the class of groups π which are $\mathcal{C}^n$ for some $n \geq 0$. $\mathcal{C}$ is not closed under sums (see [5, p. 138]).

2.2. PROPOSITION. $\mathcal{C}$ is closed under extensions.

We say that a group G is a poly($\mathcal{C}$)-group if there exists a finite sequence of subgroups

$$G = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = 1$$

such that each G_i/G_{i+1} is a $\mathcal{C}$ -group ($i = 0, \dots, n-1$).

2.3. COROLLARY. Any poly ($\mathcal{C}$)-group is a $\mathcal{C}$ -group.

For example, any poly cyclic group is a $\mathcal{C}^n$ -group, where n is the Hirsch number of the group [1]. By 2.1, any poly (PD or finite) group is a $\mathcal{C}$ -group; similarly for any poly (finitely generated abelian) group.

PROOF OF 2.2. Let $1 \rightarrow \pi_1 \rightarrow G \rightarrow \pi_2 \rightarrow 1$ be an extension of a $\mathcal{C}^n$ -group π_1 by a $\mathcal{C}^m$ -group π_2 . Consider the Lyndon-Hochschild-Serre spectral sequence with abutment $H^i(G; F)$ and E_2 -term $E_2^{p,q} = H^p(\pi_2, H^q(\pi_1, F))$ where F is a free G -module. F is also a free π_1 -module, so

$$H^q(\pi_1, F) = \begin{cases} 0 & (i \neq n), \\ Z \otimes_{Z\pi_1} F & (i = n). \end{cases}$$

But $Z \otimes_{Z\pi_1} F$ is a free π_2 -module and hence $E_2^{p,q} = 0$ unless $(p, q) = (m, n)$. Thus $H^i(G, F) = 0$ unless $i = m + n$ and (a) is satisfied.

To show (b) let $G \triangleleft G'$. Then we have the following commutative exact diagram of groups and homomorphisms:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & \pi_1 & \rightarrow & G & \rightarrow & \pi_2 \rightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \rightarrow & \pi_1 & \rightarrow & G' & \rightarrow & G'' \rightarrow 1 \\ & & & & \downarrow & & \downarrow \\ & & & & G'/G & = & G''/\pi_2 \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array}$$

It follows that $\pi_1 \triangleleft G'$ and $\pi_2 \triangleleft G'' = G'/\pi_1$. Let F' be any free G' -module. Then

$$H^{m+n}(G, F') \cong H^n(\pi_2, H^m(\pi_1, F')) \cong H^n(\pi_2, Z \otimes_{Z\pi_1} F')$$

by (b) for π_1 . But $Z \otimes_{Z\pi_1} F'$ is a free π_2 -module and hence

$$H^{m+n}(G; F') \cong Z \otimes_{Z\pi_2} (Z \otimes_{Z\pi_1} F') \cong Z \otimes_{ZG} F'$$

as $G''/\pi_2 = G'/G$ modules. The last isomorphism follows because

$$\begin{aligned} Z \otimes_{Z\pi_2} (Z \otimes_{Z\pi_1} ZG') &\cong Z \otimes_{Z\pi_2} ZG'' \cong Z(G''/\pi_2) \cong Z(G'/G) \\ &\cong Z \otimes_{ZG} Z(G'). \end{aligned}$$

This completes 2.2.

Suppose a group G contains a $\mathcal{C}^n$ -group π as a normal subgroup. Then 2.2 implies that $H^i(G, ZG) \cong H^{i-n}(G/\pi; Z(G/\pi))$ for all $i \geq 0$.

Note added in proof: R. Bieri has shown that poly (finite or duality) groups G also have the property that $H^i(G; ZG) = 0$ for all but one value of i .

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