PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

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ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

A *symplectic* manifold is a manifold of dimension $2k$ with a closed 2-form α such that α^k is nonsingular. If M^{2k} is a closed symplectic manifold, then the cohomology class of α is nontrivial, and all its powers through k are nontrivial. M also has an almost complex structure associated with α, up to homotopy.

It has been asked whether every closed symplectic manifold has also a Kaehler structure (the converse is immediate). A Kaehler manifold has the property that its odd dimensional Betti numbers are even. H. Guggenheimer claimed [1], [2] that a symplectic manifold also has even odd Betti numbers. In the review [3] of [1], Liberman noted that the proof was incomplete. We produce elementary examples of symplectic manifolds which are not Kaehler by constructing counterexamples to Guggenheimer's assertion.

There is a representation ρ of $Z \oplus Z$ in the group of diffeomorphisms of T^2 defined by

$$(1,0) \xrightarrow{\rho} \mathrm{id}, \qquad (0,1) \xrightarrow{\rho} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

where "$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$" denotes the transformation of T^2 covered by the linear transformation of $\mathbf{R}^2$. This representation determines a bundle M^4 over T^2, with fiber T^2: $M^4 = \tilde{T}^2 \times_{Z \oplus Z} T^2$, where $Z \oplus Z$ acts on $\tilde{T}^2$ by covering transformations, and on T^2 by ρ (M^4 can also be seen as $\mathbf{R}^4$ modulo a group of affine transformations). Let Ω_1 be the standard volume form for T^2. Since ρ preserves Ω_1, this defines a closed 2-form Ω_1' on M^4 which is nonsingular on each fiber. Let ρ be projection to the base: then it can be checked that $\Omega_1' + p^*\Omega_1$ is a symplectic form. (It is, in general, true that $\Omega_1'' + K\rho^*\Omega_1$ is a symplectic form, for *any* closed Ω_1'' which is a volume form for each fiber, and K sufficiently large.) But $H_1(M^4) = Z \oplus Z \oplus Z$, so M^4 is not a Kaehler manifold.

Many more examples can be constructed. In the same vein, if M^{2k} is a closed symplectic manifold, and if N^{2k+2} fibers over M^{2k} with the fundamental class of the fiber not homologous to zero in N, then N is also a symplectic manifold. If, for instance, the Euler characteristic of the fiber is not zero, this

Received by the editors July 31, 1974.
AMS (*MOS*) *subject classifications* (1970). Primary 57D15, 58H05.

hypothesis is satisfied. To do this, one must see that if there is a closed 2-form α_1 whose integral on a fiber is nonzero, then α_1 is cohomologous to a 2-form α which is nonsingular on each fiber. To find α, first find a 2-form β, not necessarily closed, which is nonsingular on each fiber, and whose integral on each fiber agrees with that of α_1: this exists by convexity considerations. On each fiber, F, there is a form γ_F such that $\beta_F - (\alpha_1)_f = d(\gamma_F)$. This equation can also be solved differentiably in a small neighborhood of the base, so, by convexity considerations, there is a global 1-form γ such that on each fiber, $\beta_F - (\alpha_1)_F = d(\gamma_F)$. Let $\alpha = \alpha_1 + d(\gamma)$. If Ω_1 is a symplectic form for M^{2k}, then $\Omega = \alpha + K(\rho^* \Omega_1)$ is a symplectic form for N^{2k+2}, K is sufficiently large.

This construction, although it applies only to a narrow range of examples, nonetheless has a certain amount of flexibility. This leads me to make the

CONJECTURE. Every closed $2k$-manifold which has an almost complex structure τ and a real cohomology class α such that $\alpha^k \neq 0$ has a symplectic structure realizing τ and α.

I would like to thank Alan Weinstein for pointing out this question and for helpful discussions.

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