UNIQUENESS OF TOPOLOGY FOR THE p -ADIC INTEGERS¹

LAWRENCE CORWIN

ABSTRACT. It is shown that the only Hausdorff topologies on $\mathbf{Z}_p$, the p -adic integers, which make it into a locally compact Abelian group are the p -adic and discrete topologies. The key ingredient in the proof is a structure theorem for certain LCA groups which may be of independent interest.

Recently Kallman proved [3] that a compact simple Lie group has a unique nondiscrete Hausdorff topology which makes it a locally compact topological group. Groups with this property seem to be fairly rare. In this paper, we show that the p -adic integers enjoy the same property.

Our main tool is a structure theorem involving unique divisibility. Let G be a locally compact Abelian group (the group operation is written additively), and let p be a prime. G is said to be uniquely divisible by p if for every $a \in G$, the equation $px = a$ has exactly one solution.

THEOREM 1. *Let S be any set of primes, and suppose that G is uniquely divisible by all $p \in S$. Then G can be written as $G_1 \times G_2 \times G_3$, where:*

- (a) G_1 is a vector group;
- (b) $G_2 = \prod_{p \in S} (G_{2,p}; K_{2,p})$, a restricted direct product of groups, where $G_{2,p}$ is isomorphic to a group $H_{A,p}$ of the following form: A is a cardinal, and $H_{A,p} \subseteq \prod_{\alpha < A} Q_p$ is the subgroup $\bigcup_{n \in \mathbf{Z}} \prod_{\alpha < A} p^{-n} \mathbf{Z}_p$ topologized by letting $\prod_{\alpha < A} \mathbf{Z}_p$ be an open subgroup $K_{2,p} \cong \prod_{\alpha < A} \mathbf{Z}_p$.
- (c) G_3 has a compact open subgroup C uniquely divisible by all $p \in S$.

NOTE. In the special case where S is the set of all primes, C is a divisible subgroup and, hence, an algebraic direct summand; it is also clear that C is a topological direct summand. Hence $G_3 = C \times G_4$, where G_4 is a divisible torsion-free discrete subgroup and, hence, a product of copies of $\mathbf{Q}$. The result in this case is a theorem of Mackey's [5].

PROOF OF THEOREM 1. We may always write $G = G_0 \times G_1$, where G_0 contains a compact open subgroup K_0 and G_1 is a vector group; see, e.g., [2]. That takes care of G_1 , and we may now worry about G_0 .

So consider K_0 and its dual $K_0^\wedge$. For $p \in S$, $x \mapsto px$ is a continuous 1-1 map from K_0 onto its image. Given $\alpha \in K_0^\wedge$, we may define a character β_0 on pK_0 by $\beta_0(px) = \alpha(x)$. Extend β_0 to a character β on K_0 (see [2, Corollary 24.12]); then $p\beta = \alpha$. It follows that the p -primary component of $K_0^\wedge$ is

Received by the editors October 23, 1974 and, in revised form, November 28, 1974.

AMS (MOS) subject classifications (1970). Primary 22B05.

Key words and phrases. Locally compact Abelian group, p -adic integers, uniquely divisible by a prime.

¹ Supported by NSF Grant GP-30673.

© 1976, American Mathematical Society

divisible (hence a direct sum of groups $\cong \mathbf{Z}(p^\infty)$ —see [4, p. 10]), and therefore a direct summand. So $\hat{K}_0 = C \oplus (\oplus_{p \in S} K_p)$, where $\hat{K}_p$ is a direct sum of groups $\cong \mathbf{Z}(p^\infty)$, and $\hat{C}$ has no p -primary component for $p \in S$. Correspondingly, we have $K_0 = C \times \prod_{p \in S} K_p$, where K_p is a product of groups all $\cong \mathbf{Z}_p$.

Now we are in business. For each $p \in S$, let $G_{2,p} = \{x \in G: p^n x \in K_p \text{ for some integer } n\}$. $G_{2,p}$ is of the desired form. In $G_{2,p}$, one can divide by p because of the construction, and by other primes because one can in K_p ; thus $G_{2,p}$ is divisible. $G_2 = \prod_{p \in S} (G_{2,p} : K_{2,p})$ is therefore a divisible subgroup of G_0 containing $\prod_{p \in S} K_p$ as a compact open subgroup. The group $G_2 \times C$ is an open subgroup of G_0 , and by [2, Theorem A-8], there is a group $G_3 \supseteq C$ such that $G_2 \times G_3 \cong G_0$ algebraically. But because $G_2 \times C$ is open, the direct product decomposition is also topological. Finally, C is compact open in G_3 , and $x \mapsto px (p \in S)$ is an automorphism of C because it is an automorphism of G_3 .

Note, as a corollary, that $x \mapsto px (p \in S)$ is actually an automorphism if $G_{2,p}$ is a finite-dimensional vector space over $\mathbf{Q}_p$ for all $p \in S$. This generalizes a result in [1].

We can now prove our main result.

THEOREM 2. *The only topologies on $\mathbf{Z}_p$ which make it a locally compact Hausdorff topological group are the usual one and the discrete topology.*

PROOF. Let G be $\mathbf{Z}_p$ equipped with its new topology. We note first that if this topology is $\cong$ the $\mathbf{Z}_p$ topology, then it is the $\mathbf{Z}_p$ topology; that is, automorphisms of $\mathbf{Z}_p$ are continuous. Since any automorphism takes $p^n \mathbf{Z}_p$ to itself, this is clear.

It suffices, therefore, to show that any nondiscrete locally compact topology on G makes $G \cong \mathbf{Z}_p$. Apply Theorem 1, with $S = \{\text{all primes } q \neq p\}$; then $G = \mathbf{R}^n \times G_2 \times G_3$, where $\mathbf{R}^n$ and G_2 are divisible. As no element of G except 0 is infinitely divisible, $n = 0$ and $G_2 = (0)$. Hence, $G = G_3$ has a compact open subgroup C uniquely divisible by all primes $\neq p$. C is not trivial, since G is nondiscrete.

Next, we show that $C \cong \mathbf{Z}_p$. Since C is torsion-free, $\hat{C}$ is divisible [2, Theorem 24.23]; since $x \mapsto qx (q \text{ prime}, q \neq p)$ is an automorphism of p , $\hat{C}$ contains no subgroup $\cong \mathbf{Z}(q^\infty)$. If $\hat{C}$ contains a subgroup $\cong \mathbf{Q}$, then we could write $C = C_0 \times C_{00}$, where $C_{00} \cong \hat{\mathbf{Q}}$ is divisible and therefore an algebraic direct summand of $\mathbf{Z}_p$. But this is impossible, since $\mathbf{Z}_p$ is indecomposable [4, p. 46]. Therefore $\hat{C}$ is a direct sum of groups $\cong \mathbf{Z}(p^\infty)$, and C is a direct product of groups $\cong \mathbf{Z}_p$. So write $C = C_1 \times C_2$, with $C_1 \cong \mathbf{Z}_p$. Our job is to show that $C_2 = (0)$.

Let $F = C_1 - pC_1$, and let $U_n = \{x \in F: \exists y \in G \text{ with } p^n y = x\}$. Since $x \in U_n \Rightarrow p^n C_1 + x \subseteq U_n$, U_n is open; also, $\bigcup_{n=1}^\infty U_n = F$ and $U_n \subseteq U_{n+1}$. From compactness, $\exists n$ with $U_n = F$. Write F as the union of $p^n C_1$ -cosets; on a typical coset, $x_j + p^n C_1$, every element is divisible by the same powers of p . Suppose that $x_j = p^j y$, but that x_j is not divisible by p^{j+1} . Then the group C_3 generated by the y_j and C_1 is a subgroup of G containing C_1 as a subgroup of finite index; it is therefore compact. Moreover, C_3 is a pure subgroup of G . From [2, Theorem 25.21], C_3 is an algebraic direct summand of G . As G is

indecomposable, $C_3 = G$. Hence $C_2 = \{0\}$, and $G/C_1 = C_3/C_1$ is finite.

Suppose $[G : C_1] = r$. Then $rG \subseteq C_1$, and so C_1 contains some $p^m \mathbf{Z}_p$. Thus C_1 is an open subgroup of $\mathbf{Z}_p$ with a topology (perhaps not the one inherited from $\mathbf{Z}_p$) making it $\cong \mathbf{Z}_p$. As noted in the first paragraph, this topology must be the usual one (note that any open subgroup of $\mathbf{Z}_p$ is topologically isomorphic to $\mathbf{Z}_p$). The theorem follows.

It is considerably easier to prove that the only topology on $\mathbf{Z}_p$ which makes it a compact group is the usual one. Let Γ be the dual group for some such topology. Since $\mathbf{Z}_p$ is torsion-free, Γ is divisible; since $\mathbf{Z}_p$ is indecomposable, $\Gamma \cong Q$ or some $\mathbf{Z}(p^\infty)$; since $\mathbf{Z}_p$ is not infinitely divisible by p and $\Gamma^\wedge$ is unless $\Gamma = \mathbf{Z}(p^\infty)$, Γ must be $\mathbf{Z}(p^\infty)$.

BIBLIOGRAPHY

1. L. Corwin, *Generalized Gaussian measures and a "functional equation"*. I, J. Functional Analysis **5** (1970), 412–427. MR **41** #4135.
2. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #158.
3. R. R. Kallman, *The topology of compact simple Lie-groups is essentially unique*, Advances in Math. **12** (1974), 416–417.
4. I. Kaplansky, *Infinite abelian groups*, Univ. of Michigan Press, Ann Arbor, Mich., 1974. MR **16**, 444.
5. G. W. Mackey, *A remark on locally compact abelian groups*, Bull. Amer. Math. Soc. **52** (1946), 940–944. MR **8**, 311.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN CONNECTICUT 06520

Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903