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# QUASI-UNMIXEDNESS AND INTEGRAL CLOSURE OF REES RINGS

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ABSTRACT. For certain Rees rings $\mathfrak{R}$ of a local domain $R$, the quasi-unmixedness of $R$ is characterized in terms of a certain transform of $\mathfrak{R}$ being contained in the integral closure of $\mathfrak{R}$.

1. **Introduction.** In this paper, a ring shall be a commutative ring with identity. The terminology is basically that of [2] and [12].

Relations between quasi-unmixedness and integral extensions are well known (e.g., [1], [5] and [7]). Also, the study of properties of a ring $R$ via transition to a Rees ring $\mathfrak{R} = \mathfrak{R}(R,A)$ of $R$ (conditions on the ideal $A$ depending on the particular discussion) has often been useful. In particular, characterizations of the quasi-unmixedness of $R$ are given in [10] in terms of localizations of $\mathfrak{R}$ containing $R$ as a quasi-subspace. The $\mathfrak{R}$-algebra $\mathfrak{T} = \mathfrak{T}(u\mathfrak{R})$ (Definition 1) is used in [8] to characterize unmixed local domains. Here, equivalences to the quasi-unmixedness of $R$ are given in terms of $\mathfrak{T}$ being contained in the integral closure of $\mathfrak{R}$ (Theorem 2).

2. **Preliminary concepts.** Let $B = (b_1, \ldots, b_k)R$ be an ideal in a Noetherian ring $R$. Let $t$ be an indeterminant, and let $u = 1/t$. The *Rees ring* $\mathfrak{R} = \mathfrak{R}(R,B)$ *of $R$ with respect to $B$* is the ring $\mathfrak{R} = R[u, tb_1, \ldots, tb_k]$. $\mathfrak{R}$ is a graded Noetherian subring of $R[u,t]$. If $(R, M)$ is a local ring, then $\mathfrak{M} = (M, u, tb_1, \ldots, tb_k)$ is the unique maximal homogeneous ideal of $\mathfrak{R}$. Similar to [12, Theorem 11, p. 157], $\mathfrak{R}'$ is a graded subring of $K[u,t]$, where $K$ is the total quotient ring of $R$. (Throughout, $S'$ will denote the integral closure of ring $S$.)

For an ideal $B$ in a ring $R$, the *integral closure* of $B$ in $R$, denoted $B_a$, is the set of all elements in $R$ satisfying an equation of the form $x^n + b_1 x^{n-1} + \cdots + b_n = 0$, where $b_i \in B^i$, $i = 1, \ldots, n$. It is known [4, p. 523] that $B_a$ is an ideal in $R$. In particular, if $B = bR$ is a regular principal ideal, then $(bR)_a = \{r \in R; r/b \in R'\} = bR' \cap R$ [6, Lemma 1].

DEFINITION 1. Let $b$ be a regular nonunit in a ring $R$. Define $\mathfrak{T}(bR) = \{c_k/b^k; c_k \in (b^k R)^{(1)}$, for all large $k\}$, where $(b^k R)^{(1)}$ is the set of elements of $R$ that are in each height one primary component of $b^k R$.

REMARK. The following are shown in [11].

(1) $\mathfrak{T}(bR)$ is contained in $R'$ if and only if each height one prime divisor of

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$bR'$ contracts to a height one prime (divisor of $bR$) in $R$.

(2) $b^n\mathfrak{T}(bR)$ is a finite intersection of height one primary ideals. Also $b^n\mathfrak{T}(bR) \cap R = (b^n R)^{(1)}$.

(3) Define $R^{(1)} = \bigcap\{R_{(P)}; P$ is a height one prime divisor of a principal ideal generated by a nonzero divisor in $R\}$, where $(P)$ denotes the set of regular elements in $R - P$. Then $\mathfrak{T}(bR) = R[1/b] \cap R^{(1)}$.

3. **Characterizations of quasi-unmixed local domains.** Several preliminary results on completions are given to show that the condition $\mathfrak{T} \subseteq \mathfrak{R}'$ is equivalent to a similar condition for the completion $R^*$ of $R$ (Corollary 1). This is used to give equivalences to the quasi-unmixedness of a local domain (Theorem 2).

LEMMA 1. *Let $B$ be an $M$-primary ideal of a local ring $(R, M)$. Let $\mathfrak{R} = \mathfrak{R}(R, B)$. Let $p$ be a prime ideal of $\mathfrak{R}$ with $u\mathfrak{R} \subseteq p$. Then $(M, u)\mathfrak{R} \subseteq p$, and so all prime ideals containing $u\mathfrak{R}$ lie over $M$.*

PROOF. Since $u$ is in $p$, $B = u\mathfrak{R} \cap R \subseteq p \cap R$. But $B$ is $M$-primary, so $M \subseteq p \cap R$, i.e., $M = p \cap R$. *Q.E.D.*

LEMMA 2. *Let $\mathfrak{R}$ be as in Lemma 1 and $\mathfrak{S} = \mathfrak{R}(R^*, BR^*)$. Let $\mathfrak{M}$ (resp., $\mathfrak{M}'$) be the maximal homogeneous ideal of $\mathfrak{R}$ (resp., $\mathfrak{S}$), and let $\mathfrak{R}^*$ (resp., $\mathfrak{S}^*$) be the completion of $\mathfrak{R}$ (resp., $\mathfrak{S}$) with respect to the $\mathfrak{M}$ (resp., $\mathfrak{M}'$)-adic topology. Then $\mathfrak{R}^* = \mathfrak{S}^*$ is the completion $(\mathfrak{R}_{\mathfrak{M}})^* = (\mathfrak{S}_{\mathfrak{M}'})^*$ of $\mathfrak{R}_{\mathfrak{M}}$ and $\mathfrak{S}_{\mathfrak{M}'}$.*

PROOF. $\mathfrak{R}_{\mathfrak{M}}$ is a dense subspace of $\mathfrak{S}_{\mathfrak{M}'}$ [8, Lemma 3.2] and $\mathfrak{R}^*$ (resp., $\mathfrak{S}^*$) is the natural completion of $\mathfrak{R}_{\mathfrak{M}}$ (resp., $\mathfrak{S}_{\mathfrak{M}'}$) [3, Theorem 32, p. 434]. Q.E.D.

LEMMA 3. *Let $R$, $R^*$, $B$, $\mathfrak{R}$ and $\mathfrak{S}$ be as in Lemma 2. Also, assume that $B$ is generated by a system of parameters. Let $\mathfrak{T} = \mathfrak{T}(u\mathfrak{R})$ and $\mathfrak{T}^* = \mathfrak{T}(u\mathfrak{S})$. Then $N = (M, u)\mathfrak{R}_{((M,u))} \cap \mathfrak{T}$ (resp., $N^* = (M^*, u)\mathfrak{R}_{((M^*,u))} \cap \mathfrak{T}^*$) is the only prime divisor of $u\mathfrak{T}$ (resp., $u\mathfrak{T}^*$).*

PROOF. By [8, Remark 3.10(ii)], $(M, u)\mathfrak{R}$ is the only height one prime divisor of $u\mathfrak{R}$. By the one-to-one correspondence (and denseness) in [8, Lemma 3.2], $(M^*, u)\mathfrak{S} = (M, u)\mathfrak{S}^*$ is the only height one prime divisor of $u\mathfrak{S}$, and by the one-to-one correspondence in [11, Lemma 2(9)], $N$ (resp., $N^*$) is the only height one prime divisor of $u\mathfrak{T}$ (resp., $u\mathfrak{T}^*$). By Remark (2), this ideal has no imbedded prime divisors. Q.E.D.

THEOREM 1. *With the notation of Lemma 2, let $p \subseteq P$ be an inclusion of prime ideals in $\mathfrak{R}$ with $u \in p$. Then the following statements hold*:

(1) *$\mathfrak{R}/p$ is a locally unmixed, pseudo-geometric domain* [2, *p.* 131].

(2) *$p\mathfrak{R}_P^*$ is a semiprime, unmixed ideal in the completion $\mathfrak{R}_P^*$ of $\mathfrak{R}_P$.*

(3) *In the completion $\mathfrak{R}_P^*$ of $\mathfrak{R}_P$, $p\mathfrak{R}_P^*$ has pure height equal to* height $p$ *and has pure depth equal to* depth $p\mathfrak{R}_P$.

(4) *$p\mathfrak{R}^* = p\mathfrak{S}^*$ has pure height equal to* height $p$, *where $p$ is contained in the maximal homogeneous ideal of $\mathfrak{R}$.*

PROOF. Since $p \cap R = M$ (Lemma 1), $\mathfrak{R}/p = (R/M)[u^\#, (tB)^\#]$, where $X^\#$ denotes $X$ modulo $p$. Thus $\mathfrak{R}/p$ is finitely generated as a ring over the field

$R/M$, and so is locally unmixed [2, (34.9)], and pseudo-geometric [2, (36.5)]. This shows (1). By localizing to $\mathfrak{R}_P$, (2) follows from [2, (36.4)] and (1).

For (3), since $p\mathfrak{R}_P^*$ is an unmixed ideal (by (2)), it has pure depth equal to depth $p\mathfrak{R}_P^* =$ depth $p\mathfrak{R}_P$. Since $p\mathfrak{R}_P^*$ is semiprime, that it has pure height equal to height $p$ follows from [2, (22.9)]. (4) is a special case of (3) since $\mathfrak{R}_{\mathfrak{M}}^* = \mathfrak{R}^* = \mathfrak{S}^*$ by Lemma 2. Q.E.D.

COROLLARY 1. *Let the notation be as in Lemma 2. Then $\mathfrak{T}(u\mathfrak{R}) \subseteq \mathfrak{R}'$ if and only if $\mathfrak{T}(u\mathfrak{S}) \subseteq \mathfrak{S}'$.*

PROOF. Since $(u^n\mathfrak{R})_a = u^n\mathfrak{R}' \cap \mathfrak{R}$ and $\mathfrak{R}'$ and $\mathfrak{R}$ are graded subrings of $K[u, t]$, it follows that $(u^n\mathfrak{R})_a$ is a homogeneous ideal in $\mathfrak{R}$. Therefore, every prime divisor of $(u^n\mathfrak{R})_a$, for $n \geqslant 1$, and every prime divisor of the homogeneous ideal $u\mathfrak{R}$ is contained in the maximal homogeneous ideal $\mathfrak{M}$ of $\mathfrak{R}$. By [11, Lemma 4(2)], $\mathfrak{T}(u\mathfrak{R}_{\mathfrak{M}}) \subseteq \mathfrak{R}'_{\mathfrak{M}}$ if and only if $\mathfrak{T}(u\mathfrak{R}) \subseteq \mathfrak{R}'$. Now, let $P$ be a height one prime divisor of $u\mathfrak{R}_{\mathfrak{M}}$, and $p = P \cap \mathfrak{R}$. Then $P\mathfrak{R}_{\mathfrak{M}}^* = P\mathfrak{R}^*$ has pure height one (Theorem 1(4)). Therefore, by [11, Corollary 2], $\mathfrak{T}(u\mathfrak{R}_{\mathfrak{M}}) \subseteq \mathfrak{R}'_{\mathfrak{M}}$ if and only if $\mathfrak{T}(u\mathfrak{R}_{\mathfrak{M}}^*) \subseteq \mathfrak{R}_{\mathfrak{M}}^{*\prime}$. But $\mathfrak{R}_{\mathfrak{M}}^* = (\mathfrak{S}_{\mathfrak{M}'})^*$ so the last inclusion is equivalent to $\mathfrak{T}(u\mathfrak{S}_{\mathfrak{M}'})^* \subseteq (\mathfrak{S}_{\mathfrak{M}'})^{*\prime}$. As above, this is equivalent to $\mathfrak{T}(u\mathfrak{S}_{\mathfrak{M}'}) \subseteq (\mathfrak{S}_{\mathfrak{M}'})'$, which, again as above, is equivalent to $\mathfrak{T}(u\mathfrak{S}) \subseteq \mathfrak{S}'$. Q.E.D.

LEMMA 4. *Let $b$ be a regular nonunit in a Noetherian ring $R$ and $q$ a minimal prime divisor of zero in $R'$. Then there exists a height one prime divisor $P$ of $bR'$ that contains $q$.*

PROOF. In $R'$, let $Z = \operatorname{rad}(0) = \bigcap_{i=1}^n q_i$ $(q_1 = q)$. Since $Z \subseteq bR'$ [9, Lemma 2.4], we may pass to $R'/Z = \overline{R}$. $\overline{R}$ is the direct sum of Krull domains $\oplus_{i=1}^n R'/q_i = \oplus_{i=1}^n \overline{R}e_i$, where the $e_i$ are the associated orthogonal idempotents. A height one prime divisor $p_1$ of $be_1$ in $\overline{R}e_1$ gives rise to the desired $P$. Q.E.D.

THEOREM 2 (cf. [8, Theorem 5.17]). *Let $(R, M)$ be a local domain of altitude $n \geqslant 1$. Then the following statements are equivalent*:

(1) *$R$ is quasi-unmixed.*

(2) *For every finitely generated domain $A$ over $R$, and for each multiplicatively closed subset $S$ of $A$, $(A_S)^{(1)} \subseteq A_S'$.*

(3) *For every ideal $B$ in $R$, $\mathfrak{T}(u\mathfrak{R}) \subseteq \mathfrak{R}'$, where $\mathfrak{R} = \mathfrak{R}(R, B)$.*

(4) *There exists an $M$-primary ideal $B$ in $R$ that is generated by a system of parameters such that $\mathfrak{T}(u\mathfrak{R}) \subseteq \mathfrak{R}'$, where $\mathfrak{R} = \mathfrak{R}(R, B)$.*

PROOF. $(1 \Rightarrow 2)$. By [11, Lemma 1(3) and (5)], it is sufficient to show $A^{(1)} \subseteq A'$. By [5, Corollary 2.5], $A$ is locally quasi-unmixed. Then, by [7, Theorem 3.8], each height one prime ideal in $A'$ contracts to a height one prime in $A$. Thus, by [8, Corollary 5.7], $A^{(1)} \subseteq A'$.

$(2 \Rightarrow 3)$. Since $\mathfrak{R}$ is a finite extension of $R$, $\mathfrak{R}^{(1)} \subseteq \mathfrak{R}'$, by hypothesis. And, $\mathfrak{T}(u\mathfrak{R}) \subseteq \mathfrak{R}^{(1)}$.

$(3 \Rightarrow 4)$ is obvious.

$(4 \Rightarrow 1)$. Let $B$ be an $M$-primary ideal of $R$ generated by a system of parameters. Let $\mathfrak{T} \subseteq \mathfrak{R}'$, where $\mathfrak{R} = \mathfrak{R}(R, B)$ and $\mathfrak{T} = \mathfrak{T}(u\mathfrak{R})$. By Corollary 1, $\mathfrak{T}^* \subseteq \mathfrak{S}'$, where $\mathfrak{S} = \mathfrak{R}(R^*, BR^*)$ and $\mathfrak{T}^* = \mathfrak{T}(u\mathfrak{S})$ ($R^*$ is the completion of

$R$). Let $q$ be a minimal prime divisor of zero in $\mathbb{S}$. Let $q'$ be the minimal prime divisor of zero in $\mathbb{S}'$ that lies over $q$ ($\mathbb{S}$ and $\mathbb{S}'$ have the same total quotient ring). By Lemma 4, there exists a height one prime divisor $p'$ of $u\mathbb{S}'$ that contains $q'$. By Remark 1, $p' \cap \mathbb{S} = p$ is a height one prime divisor of $u\mathbb{S}$. Hence, $q \subseteq p = (M^*, u)\mathbb{S}$ (Lemma 3). Since $q$ was an arbitrary minimal prime divisor of zero in $\mathbb{S}$, $R$ is quasi-unmixed [10, Corollary 9]. Q.E.D.

By combining Theorem 2 and the Remark, further characterizations of the quasi-unmixedness of $R$ can be obtained.

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