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PERTURBATIONS OF LIMIT-CIRCLE EXPRESSIONS

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ABSTRACT. It is shown that for any limit-circle expression $L(y) = \sum_{j=0}^{n} p_j y^{(j)}$, any sequence of disjoint intervals $\{[a_k, b_k]\}_{k=1}^{\infty}$ such that $a_k \to \infty$ as $k \to \infty$, and any $i \leqslant n-1$, there is an expression $M(y) = \sum_{j=0}^{n} q_j y^{(j)}$ such that $q_i = p_i$ except on $\bigcup (a_k, b_k)$, $q_j = p_j$ for all $j \neq i$, and such that M is not limit-circle.

An nth order ordinary differential expression $L(y) = \sum_{j=0}^{n} p_j y^{(j)}$, where each p_j is a complex-valued function on $[0, \infty)$ with continuous jth derivative and p_n is zero-free, is said to be *limit-circle* if all solutions of $L(y) = 0$ and all solutions of $L^+(y) = 0$ lie in $L^2(0, \infty)$. Here L^+ is the formal (Lagrange) adjoint of L. The smoothness assumptions on the p_j's ensure the existence of L^+ as a differential expression. They can be avoided by suitable use of quasi-differential expressions. See [4].

We shall show that the limit-circle property depends on the behavior of the coefficient functions on the entire interval. More precisely, we have

THEOREM. *Let $L(y) = \sum_{j=0}^{n} p_j y^{(j)}$ be limit-circle, let $\{[a_k, b_k]\}_{k=1}^{\infty}$ be any sequence of pairwise disjoint intervals such that $a_k \to \infty$ as $k \to \infty$, and let $i \leqslant n-1$. Then there is an expression $M(y) = \sum_{j=0}^{n} q_j y^{(j)}$ such that $q_i = p_i$ except on $\bigcup_{k=1}^{\infty}(a_k, b_k)$, $q_j = p_j$ for each $j \neq i$, and such that M is not limit-circle.*

REMARK. If L is a second order real formally symmetric expression, then L is either limit-circle or limit-point. Thus the theorem asserts in this case that for any limit-circle expression $-(ry')' + py$ there is a limit-point expression $-(ry')' + qy$ such that $q = p$ on the complement of a prescribed sequence of intervals. This extends, in part, a result of Eastham and Thompson [1] who show that for a certain class of limit-circle expressions the above conclusion holds with the added property that q is monotonic. Our assertion, for second order real formally symmetric expressions, can also be deduced from a limit-point criterion of Ismagilov [2] (for leading coefficient 1) and Knowles [3].

PROOF. The proof is based on the observation that a necessary (though far from sufficient) condition for L to be limit-circle is that there exist a positive constant K such that

$$\|L(f)\| \geqslant K\|f\| \tag{1}$$

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for all C^∞ functions f with compact support in the interior of $[0, \infty)$. Here $\|\cdot\|$ denotes the usual norm in $L^2(0, \infty)$. This may be seen as follows. There are solutions $\varphi_1, \ldots, \varphi_n$ of $L(y) = 0$ and $\psi_1, \ldots, \psi_n$ of $L^+(y) = 0$ such that $V(x, t) = \sum_{j=1}^n \varphi_j(x)\psi_j(t)$ has the property that for any $g \in L^2(0, \infty)$, $f(x) = \int_0^x V(x, t)g(t)\,dt$ satisfies $L(f) = g$ and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Thus a restriction of this integral operator is the inverse of the operator determined by the differential expression L on the linear space of C^∞ functions with compact support in the interior of $[0, \infty)$. From the assumption that L is limit-circle it is clear that this integral operator is a Hilbert-Schmidt operator and so, in particular, continuous. Thus the inequality (1) is valid for some positive K.

Now suppose that $\{[a_k, b_k]\}_{k=1}^\infty$ is given. We complete the proof first for $i = 0$. From the above observation it suffices to define M on $\bigcup_{k=1}^\infty [a_k, b_k]$ so that for each k there is a C^∞ function f_k supported on the interior of $[a_k, b_k]$ such that $\|M(f_k)\| < (1/k)\|f_k\|$. It will be convenient to adopt the notation $L_1(y) = \sum_{j=1}^n p_j y^{(j)} = L(y) - p_0 y$. We may consider the intervals independently and so must only show that given any interval $[a, b]$ and any $\varepsilon > 0$ there is a function q_0 on $[a, b]$ such that $q_0(a)$ and $q_0(b)$ have prescribed values (to make q_0 continuous on $[0, \infty)$), and a C^∞ function f supported on the interior of $[a, b]$ such that $M(y) = L_1(y) + q_0 y$ satisfies $\|M(f)\| < \varepsilon\|f\|$.

Let f be any fixed C^∞ function supported on $[\alpha, \beta] \subset (a, b)$ such that $f(x) > 0$ for $\alpha < x < \beta$ and $\|f\| = 1$. Choose γ and δ with $\alpha < \gamma < \delta < \beta$ so that

$$\int_\alpha^\gamma |L_1(f)|^2\,dt + \int_\delta^\beta |L_1(f)|^2\,dt < \varepsilon^2/4.$$

Define q_0 on $[\gamma, \delta]$ by $q_0 = -L_1(f)(x)/f(x)$. If q_0 is then extended first to $[\alpha, \beta]$ so that it vanishes outside a sufficiently small neighborhood of $[\gamma, \delta]$, and then to $[a, b]$ so that $q_0(a)$ and $q_0(b)$ have the prescribed values, then

$$\int_\alpha^\gamma |q_0 f|^2\,dt + \int_\delta^\beta |q_0 f|^2\,dt < \varepsilon^2/4.$$

Thus

$$\|M(f)\|^2 = \int_\alpha^\gamma |M(f)|^2\,dt + \int_\delta^\beta |M(f)|^2\,dt < \varepsilon^2,$$

and the proof is complete when $i = 0$.

For $i > 0$ the above construction may be repeated with the obvious modifications. The only additional complication is that $f^{(i)}$ will have zeros in the interior of its support. However by proper choice of f we may assume that there are only finitely many of these, so that we may define $q_i(x)$ outside the union of small neighborhoods of these points so that $M(f) = 0$, and then extend q_i to $[a, b]$ as before.

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