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THE EXISTENCE OF CONJUGATE POINTS FOR SELFADJOINT DIFFERENTIAL EQUATIONS OF EVEN ORDER[1]

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ABSTRACT. This paper presents sufficient conditions on the coefficents of $L_{2n}y = \sum_{k=0}^{n}(-1)^{n-k}(p_k y^{(n-k)})^{(n-k)}$ which insure that $L_{2n}y = 0$ has conjugate points $\eta(a)$ for all $a > 0$. The main theorem implies that $(-1)^n y^{(2n)} + py = 0$ has conjugate points $\eta(a)$ for all $a > 0$ when $\int^\infty x^\alpha p(x)\,dx = -\infty$ for some $\alpha < 2n - 1$ with no sign restrictions on $p(x)$.

We shall devote our attention to the selfadjoint, linear differential equation $L_{2n}y = 0$ where

$$L_{2n}y = \sum_{k=0}^{n}(-1)^{n-k}(p_k y^{(n-k)})^{(n-k)} \qquad (p_0(x) > 0). \tag{1}$$

The coefficients $p_k(x)$ are assumed to have continuous $n - k$ derivatives for all $x > 0$.

Many authors have studied the behavior of the solutions to $L_{2n}y = 0$ with attention given to the zeros of solutions and their derivatives. A book by Swanson [16] and a paper by Barrett [2] have good organizations of the results of various authors for L_2, L_4, and third order linear differential equations. For studies of the behavior of the more general L_{2n} from a somewhat different perspective, the Lecture Notes of Coppel [4] or Kreith [8] can be consulted. In particular, many results have been primarily motivated by the well-known paper of Leighton and Nehari [9] which considers L_4. For theorems directly related to the one obtained here, although with different emphases, the reader should also see Bradley [3], Glazman [5, pp. 95-106], Hinton [7], Lewis [10], [11], and Ridenhour [15].

Given a real number a, if there is a number $b > a$ such that $L_{2n}y = 0$ has a nontrivial solution satisfying

$$y^{(i)}(a) = 0 = y^{(i)}(b) \qquad (0 \le i \le n-1),$$

then b is called a *conjugate point of* a and the least such b is denoted by $\eta(a)$.

By examining L_{2n} for the existence or nonexistence of $\eta(a)$ for all $a > 0$ we are also examining criteria for the set of points of the negative part of the

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spectrum of certain selfadjoint extensions $\tilde{L}_{2n}$ of L_{2n} to be infinite or finite, respectively (see Glazman [5, pp. 40,95,96]). Also, the existence or nonexistence of $\eta(a)$ for all $a > 0$ is sometimes refered to as the oscillation or nonoscillation of L_{2n}, respectively.

Given n and $a > 0$ we define the *set of admissible functions* $\mathcal{A}_n(b)$ for all $b > a$ to be the set of all real-valued functions y satisfying the following properties:

(i) $y^{(k)}$ is absolutely continuous on $[a, b]$ for $k = 0, 1, \ldots, n-1$,

(ii) $y^{(n)}$ is essentially bounded on $[a, b]$, and

(iii) $y^{(k)}(a) = 0 = y^{(k)}(b)$ for $k = 0, 1, \ldots, n-1$.

For all $y \in \mathcal{A}_n(b)$ we define the *quadratic functional for* L_{2n} by

$$I(y) = \int_a^b \sum_{k=0}^{n} p_k(x) |y^{(n-k)}(x)|^2 \, dx.$$

The primary tool used in this paper is Theorem 1 which is a corollary to a theorem of Reid [13]. However, the result has been known and applied in various ways for many years. For example, it is inherent in the Courant-Weyl minimax principles as well as the classical treatment of the Rayleigh quotients.

THEOREM 1. *Given a number $a > 0$, the following statements are equivalent*:
(i) *There is no conjugate point $\eta(a)$ with respect to $L_{2n}y = 0$.*
(ii) *For all $b > a$ and $y \in \mathcal{A}_n(b)$, $I(y) > 0$ when $y \not\equiv 0$.*

The next theorem is the principal result of this paper.

THEOREM 2. *If*

$$\int^{\infty} x^{\alpha} p_n(x) \, dx = -\infty \tag{2}$$

for some number α, and

$$\int^{\infty} x^{\alpha - 2(n-k)} |p_k(x)| \, dx < \infty \tag{3}$$

for $0 \leq k \leq n-1$, then $L_{2n}y = 0$ has conjugate points $\eta(a)$ for all $a > 0$.

PROOF. By Theorem 1 it will suffice to find an admissible function y for each $a > 0$ such that $I(y) < 0$.

Let $\phi(x)$ be the $2n - 1$ degree polynomial satisfying

$$\phi^{(i)}(0) = \phi^{(i)}(1) = \phi(1) = 0 \qquad (1 \leq i \leq n-1)$$

and $\phi(0) = 1$. For a given $a > 0$ we define $y(x)$ as follows:

$$\begin{aligned} y(x) &= x^{\alpha/2} \phi((2a - x)/a), && x \in [a, 2a), \\ y(x) &= x^{\alpha/2}, && x \in [2a, b), \\ y(x) &= x^{\alpha/2} \phi((x - b)/b), && x \in [b, 2b), \end{aligned}$$

and $y(x) \equiv 0$ otherwise. Clearly, $y(x)$ is admissible.

There is a number M, independent of b, such that

$$|y^{(n-k)}(x)|^2 \leq Mx^{\alpha-2(n-k)}$$

for $0 \leq k \leq n$ and all x where $y^{(n-k)}(x)$ exist. Consequently,

$$\sum_{k=0}^{n-1} \int_a^{2b} p_k(x)|y^{(n-k)}|^2\,dx$$

is bounded, independent of b, because of (3).

By (2) there is a number β such that $t \geq \beta$ implies that

$$\sum_{k=0}^{n-1} \int_a^{2b} p_k(x)|y^{(n-k)}(x)|^2\,dx + \int_a^{2a} p_n(x)|y(x)|^2\,dx$$
$$+ \int_{2a}^{t} x^\alpha p_n(x)\,dx < 0.$$

Define

$$Q(t) = \int_\beta^t x^\alpha p_n(x)\,dx$$

and let b be the largest zero of $Q(t)$ on $[\beta, \infty)$. By integrating by parts, we obtain the equality

$$\int_b^{2b} p_n(x)|y(x)|^2\,dx = -(2/b)\int_b^{2b} Q(x)\phi((x-b)/b)\phi'((x-b)/b)\,dx$$

since $y(x) = x^{\alpha/2}\phi((x-b)/b)$ on $(b, 2b)$ and $Q(b) = 0 = \phi(1)$. By noting that

$$\phi(x) = C\int_1^x [t(t-1)]^{n-1}\,dt$$

where $\phi(0) = 1$ implies that

$$C^{-1} = -\int_0^1 [t(t-1)]^{n-1}\,dt,$$

it is easy to show that $\phi(x) \geq 0$ and $\phi'(x) \leq 0$ on $[0,1]$. Therefore, since $Q(x) \leq 0$ on (b, ∞), we know by the above equality that

$$\int_b^{2b} p_n(x)|y(x)|^2\,dx < 0.$$

This implies that $I(y) < 0$ and the proof is complete.

Corollary to Theorem 2. *If for some $\alpha < 2n - 1$*

$$\int^\infty x^\alpha p(x)\,dx = -\infty$$

then $(-1)^n y^{(2n)} + py = 0$ has a conjugate point $\eta(a)$ for all $a > 0$.

The bound on α is sharp. This follows from the well-known fact that the Euler equation $(-1)^n y^{(2n)} + cx^{-2n}y = 0$ does not have conjugate points $\eta(a)$ for all $a > 0$ when $c \geq -\sigma_n^2$ where

$$\sigma_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}.$$

If $c < -\sigma_n^2$, then the Euler equation does have conjugate points $\eta(a)$ for all $a > 0$ (see Glazman [4, pp. 96,97]). Without sign restrictions on $p(x)$, significant refinement of the above corollary is not expected. With sign restrictions on $p(x)$, refinement can be obtained as in the results of Hille [6] and Glazman [5, p. 100].

Theorem 4.2 of Reid [14, p. 105] shows that $(-1)^n y^{(2n)} + py = 0$ has conjugate points $\eta(a)$ for all $a > 0$ when $p(x) \leq 0$ and

$$\int^{\infty} t^{2n-2} p(t)\, dt = -\infty.$$

The above corollary removes the sign restriction on $p(x)$ and, also improves the result when $p(x)$ is known to be negative.

Moore [12] proved a theorem which has the above corollary when $n = 1$. His theorem is a generalization of Leighton's well-known result: the equation $L_2 y = 0$ is oscillatory when

$$\int^{\infty} (p_0(x))^{-1}\, dx = -\int^{\infty} p_1(x)\, dx = \infty.$$

Using Theorem 1, Leighton and Nehari [9] proved Theorem 2 for the special case $n = 2$ with the added restrictions that $p_1(x)$ and $p_2(x)$ be negative.

In order to further examine the sharpness of Theorem 2, the next theorem, whose proof can be found in [10], and its corollary is presented.

For L_{2n} defined in (1) let $P_k^0(x) = p_k(x)$ and for $m \geq 1$ define

$$P_k^m(x) = \int_x^{\infty} P_k^{m-1}(t)\, dt$$

when P_k^{m-1} is integrable. Also, for each $k \geq 1$ we define

$$M_k = k!\, 2^{4k-1}/(2k)!\,.$$

THEOREM 3. *Suppose that for* $k = 1, \ldots, n$ *and* $m = 0, 1, \ldots, k-1$

$$-\infty < \int_a^{\infty} P_k^m(t)\, dt < \infty.$$

If $x^k |P_k^k(x)| \leq \delta_k$ *and* $\sum_{k=1}^n \delta_k M_k = 1$ *for all* $x \geq a$, *then* $L_{2n} y = 0$ *does not have a conjugate point* $\eta(a)$.

The following corollary provides an interesting comparison to the results in Theorem 2 and its corollary when $\alpha = 2n - 1$.

COROLLARY TO THEOREM 3. *If* $p_0(x) \equiv 1$ *and for* $k = 1, \ldots, n$

$$\int_c^{\infty} x^{2k-1} |p_k(x)|\, dx < \infty,$$

then $L_{2n} y = 0$ *does not have a conjugate point* $\eta(a)$ *for some* $a > c$.

Theorem 3 and its corollary are related to Ahlbrandt's [1, p. 293] Theorem 6.1. When $p_0 = 1$ and sign conditions are added the above result is stronger.

Theorem 2 creates a question as to whether a condition similar to (2) being satisfied by one of the middle terms of L_{2n} might also yield the same results. The next theorem partially answers this question when an additional sign-type restriction is permitted on the middle term.

We define $f^-(x) = f(x)$ when $f(x) \leq 0$ and zero otherwise. The function $f^+(x)$ is defined similarly.

THEOREM 4. *Suppose that for some* α

$$\int^{\infty} x^{\alpha-2(n-k)} p_k^+(x)\, dx < \infty$$

for $k = 0, 1, \ldots, n$. *If for some* $0 < m \leq n$

$$\int^{\infty} x^{\alpha-2(n-m)} p_m^-(x)\, dx = -\infty,$$

then $L_{2n}y = 0$ *has a conjugate point* $\eta(a)$ *for all* $a > 0$.

Theorem 4 is easy to prove using Theorem 1 and the admissible function $y(x)$ defined in the proof of Theorem 2 except with a different choice of b.

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