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ON TOTALLY REAL BISECTIONAL CURVATURE

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ABSTRACT. A Kaehler manifold of dimension $\geqslant 3$ is a complex space form if and only if it has constant totally real bisectional curvature.

Goldberg and Kobayashi [2] introduced the notion of holomorphic bisectional curvature on a Kaehler manifold. It is determined by two holomorphic planes. In this paper, we consider a Kaehler manifold with constant totally real (or called antiholomorphic) bisectional curvature, which is determined by an antiholomorphic plane and its image by the complex structure. Namely it is defined by $R(X, JX; Y, JY)$ for a totally real section $\{X, Y\}$. A complex space form is a Kaehler manifold of constant holomorphic sectional curvature. It turns out that a complex space form can be characterized by having constant totally real bisectional curvature.

1. Let M be a real $2n$-dimensional Kaehler manifold with complex structure J and Riemann metric g. Let R be the curvature tensor field of M. Then we have $R(JX, JY) = R(X, Y)$ and $R(X, Y)JZ = JR(X, Y)Z$ for any vectors X, Y, Z tangent to M. We denote $R(X, Y; ZW)$ by

$$R(X, Y; Z, W) = g(R(X, Y)Z, W).$$

Then the sectional curvature of M determined by orthonormal vectors X and Y is given by $K(X, Y) = R(X, Y; Y, X)$. It is easy to see then

$$K(JX, JY) = K(X, Y), \quad K(X, JY) = K(JX, Y)$$

and

$$R(X, Y; Z, W) = R(JX, JY; Z, W) = R(X, Y; JZ, JW).$$

By a plane section we mean a 2-dimensional linear subspace of a tangent space. A plane section π is called holomorphic (respectively antiholomorphic or totally real) if $J\pi = \pi$ (respectively $J\pi$ is perpendicular to π). The sectional curvature for a holomorphic (respectively totally real) plane section is called holomorphic (respectively totally real) sectional curvature. A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. Let X be a unit vector in a holomorphic plane section, then it is clear

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that the holomorphic sectional curvature of that holomorphic plane section is $K(X,JX)$. We denote $H(X) = K(X,JX)$.

It is easy to see that orthonormal vectors X and Y span a totally real section if and only if X, Y and JY are orthonormal. Hence if we use $\{X, Y\}$ to denote the plane section spanned by orthonormal vectors X and Y, then that $\{X, Y\}$ is totally real implies that each of $\{X, JY\}$ and $\{Y, JX\}$ is also totally real.

We define the *totally real bisectional curvature* $H(X, Y)$ for a totally real section $\{X, Y\}$ by

$$H(X, Y) = -R(X,JX; Y,JY), \quad \{X, Y\} \text{ is totally real.}$$

We say M is of constant totally real bisectional curvature if $H(X, Y) =$ constant for every totally real plane section $\{X, Y\}$.

We shall prove the following:

THEOREM. *A Kaehler manifold M with* $\dim M \geqslant 3$ *is a complex space form if and only if M is of constant totally real bisectional curvature.*

2. By the first Bianchi identity we have

$$\begin{aligned} H(X, Y) &= -R(X,JX; Y,JY) = R(X, Y; JY,JX) + R(X,JY; JX, Y) \\ &= R(X, Y; Y,X) + R(X,JY; JY,X) \\ &= K(X, Y) + K(X,JY). \end{aligned}$$

Hence if M is of constant totally real sectional curvature then M is of constant totally real bisectional curvature.

Conversely we assume that $H(X, Y) = -R(X,JX; Y,JY) = C$, C being a constant. Then

$$C = K(X, Y) + K(X,JY). \tag{2.1}$$

Since $\{X, Y\}$ is totally real, the plane section $\{\sqrt{2}^{-1}(X + Y), \sqrt{2}^{-1}(JX - JY)\}$ is also totally real. Thus we have

$$\tfrac{1}{4}R(X + Y, JX + JY; JX - JY, -X + Y) = -C.$$

After expanding the left side of the above relation and some cancellation we have

$$\begin{aligned} &\tfrac{1}{4}R(X + Y, JX + JY; JX - JY, -X + Y) \\ &\quad = \tfrac{1}{4}\{R(X,JX; X,JX) + R(Y,JY; Y,JY) + 2R(X,JX; Y,JY) \\ &\qquad\qquad - 4R(X,JY; X,JY)\} \\ &\quad = \tfrac{1}{4}\{-H(X) - H(Y) - 2C + 4K(X,JY)\}. \end{aligned}$$

And hence

$$H(X) + H(Y) - 4K(X,JY) = 2C. \tag{2.2}$$

We can replace Y by JY in (2.2). Noticing that $H(JY) = H(Y)$ we have

$$H(X) + H(Y) - 4K(X, Y) = 2C. \tag{2.3}$$

From (2.1), (2.2) and (2.3) we conclude that

$$H(X) + H(Y) = 4C, \quad K(X, Y) = K(X, JY) = \tfrac{1}{2}C.$$

Hence M has constant totally real sectional curvature. Thus M is of constant totally real bisectional curvature if and only if M is of constant totally real sectional curvature. On the other hand, Chen and Ogiue in [1] proved that if $\dim M \geqslant 3$, M has constant totally real sectional curvature if and only if M is a complex space form. The theorem is thus proved.

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