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STONE-ČECH COMPACTIFICATIONS VIA ADJUNCTIONS

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ABSTRACT. The Stone-Čech compactification of a space X is described by adjoining to X continuous images of the Stone-Čech growths of a complementary pair of subspaces of X. The compactification of an example of Potoczny from [P] is described in detail.

The Stone-Čech compactification of a completely regular space X is a compact Hausdorff space βX in which X is dense and C^*-embedded, i.e. every bounded real-valued mapping on X extends to βX. Here we describe βX in terms of the Stone-Čech compactification of one or more subspaces by utilizing adjunctions and completely regular reflections. All spaces mentioned will be presumed to be completely regular.

If A is a closed subspace of X and f maps A into Y, then the *adjunction space* $X \cup_f Y$ is the quotient space of the topological sum $X \oplus Y$ obtained by identifying each point of A with its image in Y. We modify this standard definition by allowing A to be an arbitrary subspace of X and by requiring f to be a C^*-embedding of A into Y.

The *completely regular reflection* of an arbitrary space Y is a completely regular space ρY which is a continuous image of Y and is such that any real-valued mapping on Y factors uniquely through ρY. The underlying set of ρY is obtained by identifying two points of Y if they are not separated by some real-valued mapping on Y. The resulting set has the property that for each real-valued mapping f on Y, a unique real-valued function $\rho(f)$ can be defined on ρY that factors f through ρY. The topology on ρY is taken to be the weakest topology so that all of the functions $\rho(f)$ so obtained are continuous.

LEMMA 1. *If A is a subspace of X and f is a C^*-embedding of A into Y, then X is C^*-embedded in $\rho(X \cup_f Y)$.*

PROOF. The mappings required in the proof are illustrated in the diagram. The mappings p_1 and p_2 are the compositions of the quotient map on $X \oplus Y$ with the embeddings of X and Y into $X \oplus Y$ and k is any real-valued mapping on X. We show that both p_2 and $\rho|p_2[X]$ are embeddings. Since f is an embedding, p_2 is one-to-one. To show that p_2 is open onto its range, it is

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sufficient to show that the image of a cozero-set of X is a cozero-set of $p_2[X]$. Since A is C^*-embedded in Y, if k is any bounded, real-valued mapping on X, $k|A$ has an extension l to Y. It then follows from the construction of $X \cup_f Y$ that a mapping g exists so that $g \circ p_2 = k$. Hence, the image of the cozero-set of k is the trace on $p_2[X]$ of the cozero-set of g, and p_2 is not only an embedding, but is additionally a C^*-embedding.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "p_2"] \arrow[ddr, bend left, "k"] & \\
Y \arrow[r, "p_1"'] \arrow[drr, bend right=40, "l"'] & X \cup_f Y \arrow[d, "\rho"'] \arrow[dr, "g"] & \\
& \rho(X \cup_f Y) \arrow[r, "\rho(g)"'] & \mathbf{R}
\end{tikzcd}$$

Since any two points of $p_2[X]$ are separated by a mapping such as g, their images in $\rho(X \cup_f Y)$ are separated by $\rho(g)$ so that ρ is one-to-one on $p_2[X]$. In addition, $\rho|p_2[X]$ is seen to send cozero-sets to cozero-sets in exactly the same manner as for p_2. Since $\rho(g) \circ \rho \circ p_2 = k$, we have shown that X is C^*-embedded in $\rho(X \cup_f Y)$. □

We will always take the embedding f in the lemma to be the embedding $\eta_A : A \to \beta A$. To shorten notation, we write $(X|\beta A)$ for $X \cup_{\eta_A} \beta A$.

THEOREM 1. *If A is a closed subspace of X such that every noncompact closed set of X meets A, then $\beta X = \rho(X|\beta A)$.*

PROOF. From the lemma, X is C^*-embedded in $\rho(X|\beta A)$ and X is easily seen to be dense in $\rho(X|\beta A)$ since A is dense in βA. We will show that $\rho(X|\beta A)$ is compact by showing that $(X|\beta A)$ is compact. Let $\mathfrak{U}$ be an ultrafilter on $(X|\beta A)$. If any closed subset belonging to $\mathfrak{U}$ is contained in $X \backslash A$, then the hypothesis on A shows that $\mathfrak{U}$ contains a compact set and therefore converges. If no closed member of A is contained in $X \backslash A$, then the family $\{(\mathrm{cl}\ S) \cap \beta A : S \in \mathfrak{U}\}$ is a filter on βA and therefore clusters to a point in βA. Hence, $(X|\beta A)$ is compact and therefore its continuous image $\rho(X|\beta A)$ is also. □

EXAMPLE 1. The space Ψ described in Exercise 5I of **[G-J]** is one interesting example where Theorem 1 applies. Construction of Ψ begins by obtaining a maximal, infinite, almost disjoint family $\mathfrak{E}$ of infinite subsets of the countable discrete space **N**. A point is added to **N** for each E in $\mathfrak{E}$ with neighborhoods of the added point being required to contain all but finitely many points of E. The set of added points is taken to be the closed subspace A in the theorem. Since $|A| = c$, $|\beta A| = 2^{2^c}$. However, Ψ is separable, so that $|\beta\Psi| = 2^c$.

Hence, the formation of $\rho(X|\beta A)$ must identify points. This is to be expected, since Ψ fails to be normal, making it unlikely that the adjunction space $(X|\beta A)$ is Hausdorff. Since the proof of Theorem 1 shows $(X|\beta A)$ to be compact, we see that the operation of forming $\rho(X|\beta A)$ is simply one of identifying points to make the compactification Hausdorff.

The theorem also applies to the more complex space described by Burke in **[B]**. That space is constructed along the lines of Ψ, with a countable product of two point discrete spaces replacing $\mathbf{N}$.

The restriction on A limits the application of Theorem 1. By using Lemma 1 twice, a more general result is obtained.

THEOREM 2. *If A is any closed subspace of X, $\beta X = \rho(\rho(X|\beta A)|\beta(X\setminus A))$.*

PROOF. Applying Lemma 1 twice, we see that X is C^*-embedded in $\rho(X|\beta A)$ and that $\rho(X|\beta A)$ is in turn C^*-embedded in $\rho(\rho(X|\beta A)|\beta(X\setminus A))$. The density of X follows easily from that of A in βA and $X\setminus A$ in $\beta(X\setminus A)$. To show compactness, let $\mathfrak{U}$ be an ultrafilter on $(\rho(X|\beta A)|\beta(X\setminus A))$. Since $\mathfrak{U}$ contains either $\rho(\beta A)$ or $\beta(X\setminus A)$, $\mathfrak{U}$ must converge. Hence, $(\rho(X|\beta A)|\beta(X\setminus A))$ and its continuous image $\rho(\rho(X|\beta A)|\beta(X\setminus A))$ are compact. □

EXAMPLE 2. To illustrate the theorem, we first consider $\mathbf{R}$ with $A = [0,\infty)$. The first adjunction $\rho(\mathbf{R}|\beta A)$ adds the "right end" of $\beta\mathbf{R}$. $\beta(\mathbf{R}\setminus A)$ is a copy of $\beta\mathbf{R}$, and in the formation of the second adjunction, the points of the right end of $\beta(\mathbf{R}\setminus A)$ are all identified with 0.

EXAMPLE 3. Using Theorem 2, the Stone-Čech compactification of the example given by Potoczny can be described. Following the notation of **[P]**, let $W = \{\lambda: \lambda < \omega_1\}$ denote the set of countable ordinals and let $T = \{(\gamma,\lambda): 0 \leqslant \gamma < \lambda < \omega_1\}$. Define a topology on $X = W \cup T$ as follows: Points of T are isolated and V is a neighborhood of a point σ of W if V contains σ and all but finitely many points of the set $T_\sigma \equiv \{(\sigma,\lambda): \lambda > \sigma\} \cup \{(\lambda,\sigma): \lambda < \sigma\}$. It follows easily that X is Hausdorff, has a base of clopen sets, is locally compact, and completely regular.

We describe βX by examining the two adjunction steps indicated by Theorem 2 where W is taken to be A. Since W is a discrete subspace of cardinality $\aleph_1$, $|\beta W| = 2^{2^{\aleph_1}}$. We will show that in the formation of $\rho(X|\beta W)$, all of the points of the "growth" $W^* = \beta W\setminus W$ are identified. The following key property of X was demonstrated in **[P]** to show that X is not even weakly normal:

(a) If F is a countably infinite subset of W, E is an un uncountable subset of W, and U and V are open subsets of X containing F and E, respectively, then $U \cap V \neq \varnothing$.

A *uniform ultrafilter* on an infinite set of cardinality η is an ultrafilter whose every member also has cardinality η. In **[H]**, Hindman shows that such a set admits 2^{2^η} uniform ultrafilters. For a point p belonging to the growth of a discrete space D let A^p denote the corresponding free ultrafilter on D. We now show that:

(b) If p is any point of W^* and q is any point of W^* corresponding to a uniform ultrafilter, then p and q are identified in $\rho(X|\beta W)$: Since $\rho(X|\beta W)$ is Hausdorff, p and q must be identified if they fail to have disjoint neighborhoods in $(X|\beta W)$. The traces on X of such a pair of neighborhoods must contain disjoint members P and Q of A^p and A^q, respectively, as subsets of W. Let F be any countable subset of P and let $E = Q$. Then F and E satisfy the conditions of (a), and thus are not contained in disjoint open subsets of X. Hence, p and q cannot have disjoint neighborhoods in $(X|\beta W)$, and are identified in $\rho(X|\beta W)$.

Thus, the formation of $\rho(X|\beta W)$ adds only a single point to X, call it ∞. Since the pre-image in $(X|\beta W)$ of a neighborhood U of ∞ must be a neighborhood of every point of W^*, U must contain all but finitely many points of W together with a neighborhood in X of each point of W included. Hence, U must also include all but finitely many points of T_σ for all but finitely many σ's in W. Call such a subset of T *doubly cofinite*. The construction of βX is completed by adjoining βT to $\rho(X|\beta W)$ and taking the reflection. In order to describe the identification of points which occurs in taking the reflection of $(\rho(X|\beta W)|\beta T)$, we first classify the free ultrafilters on T, and therefore the points of T^*, into three types. Let p belong to T^* and let A^p be the corresponding free ultrafilter on T. Then we classify A^p as follows:

Type I: A^p contains a member Z such that $|Z \cap T_\sigma| < \aleph_0$ for all σ in W.

Such ultrafilters must exist since any ultrafilter which contains the set $\{(\sigma, \sigma+1): \sigma < \omega_1\}$ is of this type.

Type II: A^p is not of Type I and A^p contains a member Z such that $|Z \cap T_\sigma| \geqslant \aleph_0$ for only finitely many σ in W.

Ultrafilters of this type must exist since any ultrafilter containing T_σ for some σ has this property.

Type III: A^p is not of either Type I or II, i.e. for every Z in A^p, $|Z \cap T_\sigma| \geqslant \aleph_0$ for infinitely many σ in W.

The existence of Type III ultrafilters follows from the following result found in **[H]**:

LEMMA 2. *If an infinite collection $\mathfrak{A}$ of subsets of the infinite discrete space D of cardinality η satisfies*:

(1) $|A| = \eta$ *for all* A *in* $\mathfrak{A}$, *and*

(2) $|A_1 \cap A_2| < \eta$ *for* A_1 *and* A_2 *distinct members of* $\mathfrak{A}$,

then there exists a uniform ultrafilter A^p *on* D *such that for each* Z *in* A^p, $|A \in \mathfrak{A}: |Z \cap A| = \eta\}| = |\mathfrak{A}|$.

Applying the lemma to the family $\mathfrak{A} = \{T_\sigma: \sigma < \omega_1\}$ shows the existence of ultrafilters of Type III.

The description of βX is completed by describing the identifications which take place in forming $\rho(\rho(X|\beta W)|\beta T)$. From the proof of Theorem 2, $(\rho(X|\beta W)|\beta T)$ is compact, so that ρ is actually the quotient map which identifies pairs of points which are not separated by open sets. If A^p is of Type I, then a straightforward case-by-case argument shows that p can be separated from any other point of $\rho(X|\beta W)$ or T^* by disjoint neighborhoods, so that such points are not identified with any other point.

If A^p is of Type II, then there is a member Z of A^p and a finite subset F of W such that $|Z \cap T_\sigma| \geqslant \aleph_0$ only for σ in F. Thus, we can write Z as follows:

$$Z = (\cup\{Z \cap T_\sigma: \sigma \in F\}) \cup (Z \backslash \cup\{T_\sigma: \sigma \in F\}).$$

The set $Z\backslash\cup\{T_\sigma: \sigma \in F\}$ cannot belong to A^p since A^p is not of Type I.

Hence $\bigcup\{Z \cap T_\sigma : \sigma \in F\}$ is in A^p. Therefore, $Z \cap T_\sigma$ belongs to A^p for some σ_0. Thus, p cannot be separated from σ_0. Since it is easily seen that p can be separated from any other point, p is identified with σ_0 in $\rho((X|\beta W)|\beta T)$.

Finally, if A^p is of Type III, every member of A^p meets T_σ in an infinite set for infinitely many σ in W. Hence, p cannot be separated from ∞ by disjoint open sets. However, p can be separated from each σ in W since for any Z in A^p, the set $Z \backslash T_\sigma$ must belong to A^p. Therefore, p is identified with ∞.

To complete the description of βX, it remains to describe the neighborhoods of the Type I points and of ∞. If A^p is of Type I, then A^p contains sets which are clopen in $\rho(X|\beta W)$ and can include only Type I points in their closures in βT. Hence, a basic neighborhood of p in βX is identical with a basic neighborhood of p in βT. We have already seen that a neighborhood of ∞ in $\rho(X|\beta W)$ must contain all but finitely many points of W together with a doubly cofinite subset of T. Since every Type III point is identified with ∞, the pre-image in $(\rho(X|\beta W)|\beta T)$ of any neighborhood of ∞ must include a neighborhood of every Type III point. To relate Type III points to doubly cofinite subsets of T, we make the following observation:

(c) If S is a subset of T, then every Type III point is contained in $\mathrm{cl}_{\beta T} S$ if and only if S is doubly cofinite: We prove the contrapositives. If $\mathrm{cl}_{\beta T} S$ fails to contain a Type III point p, then $T \backslash S$ belongs to A^p and must have an infinite intersection with infinitely many of the T_σ's. Hence, S is not doubly cofinite. Conversely, if S fails to be doubly cofinite, then $N_\sigma = (T \backslash S) \cap T_\sigma$ is infinite for all σ belonging to an infinite index set I. By applying Lemma 2 to the family $\{N_\sigma : \sigma \in I\}$, we obtain an ultrafilter on $\bigcup\{N_\sigma : \sigma \in I\}$ which can be extended to a Type III ultrafilter A^p on T. Since A^p contains $T \backslash S$, p has a βT-neighborhood which misses S.

This leads to the following property.

(d) The point ∞ has a clopen neighborhood base in βX: Let U be any closed βX-neighborhood of ∞. Then the set $F = W \backslash \mathrm{int}(U)$ is finite. Put $S = (U \backslash (\bigcup\{T_\sigma : \sigma \in F\})) \cap T$. Then S is doubly cofinite, and $\mathrm{cl}_X S = S \cup (W \backslash F)$ is clopen in X and contained in U. Hence, $\mathrm{cl}_{\beta X} S = \mathrm{cl}_{\beta X}(\mathrm{cl}_X S)$ is a clopen subset of βX, contains ∞, and is a subset of U. Hence, ∞ has a clopen base in βX.

Since it is easy to see that every other point of βX has a clopen neighborhood base, we have shown that

(e) βX is zero-dimensional, or equivalently, X is strongly zero-dimensional:

Here, by strongly zero-dimensional, we mean that disjoint zero-sets of X are separated by clopen sets. Since this is precisely the class of spaces for which the 2-compactification and the Stone-Čech compactification coincide, we have $\zeta X = \beta X$. Finally, we describe the Hewitt-Nachbin realcompactification υX of X.

(f) $\upsilon X = \rho(X|\beta W)$: The inclusion of ∞ in υX follows from the observations that every G_δ containing ∞ meets X and that υX consists of those points of βX which cannot be separated from X by G_δ's. The exclusion of Type I points follows from the fact that no realcompact space can be C-embedded and dense in a large space. Since for each Type I point p, A^p includes a C-embedded, discrete subspace of X, p cannot belong to υX.

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