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PETERSON-STEIN FORMULAS IN THE ADAMS SPECTRAL SEQUENCE

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ABSTRACT. The purpose of this note is to establish Peterson-Stein formulas for second order differentials in the Adams spectral sequence.

In [4], Peterson and Stein related certain values of secondary cohomology operations to the values of certain functional primary operations. Their method was that of universal example. Here using a different approach, we establish analogous formulas for second order and functional first order differentials which occur in the version of the Adams spectral sequence that is formulated with respect to E_* homology in the stable homotopy category [1, p. 238]. (For these formulas we do not need the assumptions which serve to identify E_2 or assure convergence.)

We begin by recalling the definition of the Adams differential. In the spectral sequence for a pair of spectra (X, Y) the group $E_r^{s,t}$ is a subquotient of $[X, E \wedge (C(i))^s \wedge Y]_t$ and $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ is induced by the additive relation

$$(i_{s+r})_*[(p_{s+r-1})_* \circ \cdots \circ (p_{s+1})_*]^{-1}(j_s)_*.$$

Here $C(i)$ denotes the mapping cone of $i\colon S^0 \to E$, W^s denotes the s-fold smash product of W with itself, and

$$i_k\colon (C(i))^k \wedge Y \to E \wedge (C(i))^k \wedge Y,$$

$$j_k\colon E \wedge (C(i))^k \wedge Y \to (C(i))^{k+1} \wedge Y$$

and $p_k\colon (C(i))^{k+1} \wedge Y \to (C(i))^k \wedge Y$ are induced by smashing with the appropriate maps from $S^0 \to E \to C(i) \to S^0$.

Now let $f\colon Y \to Y'$ be a map in our category. Then we have

PROPOSITION (P-S1). *Let $u \in [X, E \wedge Y]_0$ belong to* $\ker d_1 \cap \ker f_*$. *Then*

$$f_* d_2^0(u) = d_1^1((d_1^0)_f(u))$$

in $[X, E \wedge (C(i))^2 \wedge Y']_1 / f_*(\operatorname{im} d_1^1)$.

Dually we have

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PROPOSITION (P-S2). *Let* $u \in [X, E \wedge Y]_0$ *belong to* $\ker(d_1^0 f_*)$. *Then*

$$(d_1^1)_f(d_1^0(u)) = d_2^0(f_*(u))$$

in $[X, E \wedge (C(i))^2 \wedge Y']_1/(\operatorname{im} d_1^1 + \operatorname{im} f_*)$.

The functional differentials here are formed the usual way via the Puppe sequence.

We give a proof of P-S2, the proof of P-S1 being dual in a certain sense. The argument depends on the "simultaneous solution" to two coextension problems which is given in the

LEMMA. *Let* $X \xrightarrow{z} A \wedge B \xrightarrow{f \wedge g} C \wedge D$ *be a null-homotopic composition of maps of spectra. Then there are maps*

$$z_1 \colon X \to C(1_C \wedge g) \quad \textit{and} \quad z_2 \colon X \to C(f \wedge 1_D)$$

of degree $+1$ *so that*

(a) $Q(1_C \wedge g)z_1 \simeq (f \wedge 1_B)z$ *and* $Q(f \wedge 1_D)z_2 \simeq (1_A \wedge g)z$;

(b) *the compositions*

$$X \xrightarrow{z_1} C(1_C \wedge g) \to C(f) \wedge C(g)$$

$$X \xrightarrow{z_2} C(f \wedge 1_D) \to C(f) \wedge C(g)$$

are homotopic.

In the above and what follows we adopt the following notation for the Puppe sequence:

$$W \xrightarrow{h} Z \xrightarrow{P(h)} C(h) \xrightarrow{Q(h)} W.$$

(We also note that this lemma corrects Proposition 5 of [3] from which Propositions 1 and 2 now follow except the minus signs in their statements disappear.)

PROOF OF LEMMA. There are coextensions z_1, z_2 and z' of $(f \wedge 1_B)z$, $(1_A \wedge g)z$ and z which are formed in the obvious way so that the following diagram

$$\begin{tikzcd}
 & C(f \wedge 1_D) \arrow[dr] & \\
S^1 \wedge X \arrow[ur, "z_2"] \arrow[r, "z'"] \arrow[dr, "z_1"'] & C(f \wedge g) \arrow[r] & C(f) \wedge C(g) \\
 & C(1_C \wedge g) \arrow[ur] &
\end{tikzcd}$$

is homotopy commutative. (The unmarked arrows designate the obvious maps.)

PROOF OF P-S2. For convenience we display the maps defining the differentials occurring in the formula.

$$
\begin{tikzcd}
& & {[X, E\wedge Y]_0} \arrow[r, "(1\wedge f)_*"] \arrow[d, "(P(i)\wedge 1)_*"] \arrow[ddl, "d_1^0"'] \arrow[dr, phantom, "C"] & {[X,E\wedge Y']_0} \arrow[d, "(P(i)\wedge 1)_*"] \arrow[dddd, bend left, "d_2^0"] \\
& & {[X, C(i)\wedge Y]_0} \arrow[r, "(1\wedge f)_*"] \arrow[dl, "(i\wedge 1\wedge 1)_*"] \arrow[d, phantom, "C"] & {[X, C(i)\wedge Y']_0} \arrow[dl] \\
{[X,E\wedge C(i)\wedge C(f)]_1} \arrow[r, "(1\wedge 1\wedge Q(f))_*"] \arrow[d, "(P(i)\wedge 1\wedge 1)_*"] \arrow[dd, bend right, "d_1^1"'] & {[X,E\wedge C(i)\wedge Y]_0} \arrow[r] & {[X,E\wedge C(i)\wedge Y']_0} & \\
{[X,(C(i))^2\wedge C(f)]_1} \arrow[d] & & & {[X,(C(i))^2\wedge Y']_1} \arrow[lll, "(1\wedge 1\wedge P(f))_*"'] \arrow[uu, "(Q(i)\wedge 1\wedge 1)_*"] \arrow[d, "(i\wedge 1\wedge 1)_*"] \arrow[dlll, phantom, "C"] \\
{[X,E\wedge(C(i))^2\wedge C(f)]_1} & & & {[X,E\wedge(C(i))^2\wedge Y']_1} \arrow[lll, "1\wedge 1\wedge P(f)_*"']
\end{tikzcd}
$$

In this diagram the quadrilaterals marked with a "C" are commutative.

Now apply the lemma to $z = (P(i) \wedge 1)_*(u)$ and $f \wedge g = i \wedge (1 \wedge f)$: $S^0 \wedge (C(i) \wedge Y) \to E \wedge (C(i) \wedge Y')$. Then there are elements

$$z_1 \in [X, E \wedge C(i) \wedge C(f)]_1 \quad \text{and} \quad z_2 \in [X, (C(i))^2 \wedge Y']_1$$

so that

$$(1 \wedge 1 \wedge Q(f))_*(z_1) = d_1^0(u)$$

and

$$(Q(i) \wedge 1 \wedge 1)_*(z_2) = (P(i) \wedge 1)_*(1 \wedge f)_*(u)$$

and

$$(P(i) \wedge 1 \wedge 1)_*(z_1) = (1 \wedge 1 \wedge P(f))_*(z_2).$$

Then $(i \wedge 1 \wedge 1)_*(z_2)$ represents both $d_2^0(f_*(u))$ and $(d_1^1)_f(d_1^0(u))$ according to their definitions.

The indeterminacy in the formula is simply the larger of the indeterminacies of the two operations.

In a future paper involving Browder's work on the Kervaire invariant problem [2], we make strong use of P-S2.

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