PETERSON-STEIN FORMULAS IN THE ADAMS SPECTRAL SEQUENCE

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ABSTRACT. The purpose of this note is to establish Peterson-Stein formulas for second order differentials in the Adams spectral sequence.

In [4], Peterson and Stein related certain values of secondary cohomology operations to the values of certain functional primary operations. Their method was that of universal example. Here using a different approach, we establish analogous formulas for second order and functional first order differentials which occur in the version of the Adams spectral sequence that is formulated with respect to E_* homology in the stable homotopy category [1, p. 238]. (For these formulas we do not need the assumptions which serve to identify E_2 or assure convergence.)

We begin by recalling the definition of the Adams differential. In the spectral sequence for a pair of spectra (X, Y) the group $E_r^{s,t}$ is a subquotient of $[X, E \wedge (C(i))^s \wedge Y]_t$ and $d_r^{s,t} : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is induced by the additive relation

$$(i_{s+r})_*[(p_{s+r-1})_* \circ \cdots \circ (p_{s+1})_*]^{-1}(j_s)_*.$$

Here C(i) denotes the mapping cone of $i: S^0 \to E$, W^s denotes the s-fold smash product of W with itself, and

$$i_k \colon (C(i))^k \wedge Y \to E \wedge (C(i))^k \wedge Y,$$

$$j_k : E \wedge (C(i))^k \wedge Y \rightarrow (C(i))^{k+1} \wedge Y$$

and $p_k: (C(i))^{k+1} \wedge Y \to (C(i))^k \wedge Y$ are induced by smashing with the appropriate maps from $S^0 \to E \to C(i) \to S^0$.

Now let $f: Y \to Y'$ be a map in our category. Then we have

PROPOSITION (P-S1). Let $u \in [X, E \land Y]_0$ belong to ker $d_1 \cap \ker f_*$. Then

$$f_{\star} d_2^0(u) = d_1^1((d_1^0)_f(u))$$

in
$$[X, E \wedge (C(i))^2 \wedge Y']_1/f_{\pm} (\text{im } d_1^1).$$

Dually we have

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PROPOSITION (P-S2). Let $u \in [X, E \land Y]_0$ belong to $\ker(d_1^0 f_*)$. Then

$$(d_1^1)_f(d_1^0(u)) = d_2^0(f_*(u))$$

$$in [X, E \wedge (C(i))^2 \wedge Y']_1/(im d_1^1 + im f_*).$$

The functional differentials here are formed the usual way via the Puppe sequence.

We give a proof of P-S2, the proof of P-S1 being dual in a certain sense. The argument depends on the "simultaneous solution" to two coextension problems which is given in the

LEMMA. Let $X \xrightarrow{z} A \wedge B \xrightarrow{f \wedge g} C \wedge D$ be a null-homotopic composition of maps of spectra. Then there are maps

$$z_1: X \to C(1_C \land g)$$
 and $z_2: X \to C(f \land 1_D)$

of degree + 1 so that

(a)
$$Q(1_C \wedge g)z_1 \simeq (f \wedge 1_R)z$$
 and $Q(f \wedge 1_R)z_2 \simeq (1_A \wedge g)z$;

(b) the compositions

$$X \xrightarrow{z_1} C(1_C \land g) \rightarrow C(f) \land C(g)$$

$$X \xrightarrow{z_2} C(f \wedge 1_D) \to C(f) \wedge C(g)$$

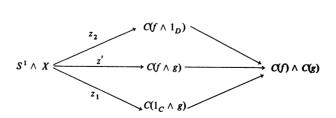
are homotopic.

In the above and what follows we adopt the following notation for the Puppe sequence:

$$W \xrightarrow{h} Z \xrightarrow{P(h)} C(h) \xrightarrow{Q(h)} W.$$

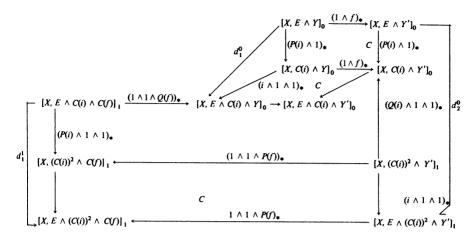
(We also note that this lemma corrects Proposition 5 of [3] from which Propositions 1 and 2 now follow except the minus signs in their statements disappear.)

PROOF OF LEMMA. There are coextensions z_1 , z_2 and z' of $(f \wedge l_B)z$, $(l_A \wedge g)z$ and z which are formed in the obvious way so that the following diagram



is homotopy commutative. (The unmarked arrows designate the obvious maps.)

PROOF OF P-S2. For convenience we display the maps defining the differentials occurring in the formula.



In this diagram the quadrilaterals marked with a "C" are commutative.

Now apply the lemma to $z = (P(i) \land 1)_*(u)$ and $f \land g = i \land (1 \land f)$: $S^0 \land (C(i) \land Y) \rightarrow E \land (C(i) \land Y')$. Then there are elements

$$z_1 \in [X, E \land C(i) \land C(f)]_1$$
 and $z_2 \in [X, (C(i))^2 \land Y']_1$

so that

$$(1 \wedge 1 \wedge Q(f))_*(z_1) = d_1^0(u)$$

and

$$(Q(i) \wedge 1 \wedge 1)_{*}(z_{2}) = (P(i) \wedge 1)_{*}(1 \wedge f)_{*}(u)$$

and

$$(P(i) \wedge 1 \wedge 1)_{*}(z_{1}) = (1 \wedge 1 \wedge P(f))_{*}(z_{2}).$$

Then $(i \wedge 1 \wedge 1)_*(z_2)$ represents both $d_2^0(f_*(u))$ and $(d_1^1)_f(d_1^0(u))$ according to their definitions.

The indeterminacy in the formula is simply the larger of the indeterminacies of the two operations.

In a future paper involving Browder's work on the Kervaire invariant problem [2], we make strong use of P-S2.

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