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FINITE OPERATORS AND AMENABLE C^*-ALGEBRAS

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ABSTRACT. In this paper we prove that the C^*-algebra generated by the left regular representation of a discrete group is amenable if and only if the group is amenable. Theorems concerning finite operators and the relationship between finite operators and amenable C^*-algebras are proved.

1. **Introduction.** Amenable C^*-algebras were introduced by B. Johnson [6]. Johnson proved that UHF and GCR C^*-algebras are amenable but left open the question of the existence of nonamenable C^*-algebras [6, 10.2, p. 91]. In §2 of this paper we prove that the C^*-algebra $C_r^*(G)$ generated by the left regular representation of a discrete group G is amenable if and only if the group G is amenable, hence exhibiting many nonamenable C^*-algebras. In §3 we prove that if T is an operator such that $C_r^*(G) \subseteq C^*(T) \subseteq C_r^*(G)''$, where $C^*(T)$ is the C^*-algebra generated by T and the identity, then G is amenable if and only if T is a finite operator in the sense of J. Williams [15]. §4 is concerned with finite operators.

2. **Amenable algebras.** Let A be a C^*-algebra. A complex Banach space X is called a Banach A-module if X is a two-sided A-module and the bilinear maps $(a,x) \to ax$ and $(a,x) \to xa$ from $A \times X$ to X are bounded. If X is a Banach A-module, then the dual space X^* becomes a Banach A-module if we define for $a \in A, f \in X^*$ and $x \in X$, $(af)(x) = f(xa)$ and $(fa)(x) = f(ax)$. A derivation from A into X^* is a linear map $D\colon A \to X^*$ such that

$$D(ab) = aD(b) + D(a)b$$

for all $a,b \in A$. By the results of J. Ringrose the derivation D is automatically norm continuous [9]. If $f \in X^*$, the function $\delta(f)\colon A \to X^*$ given by $\delta(f)(a) = af - fa$ is called the inner derivation induced by f. A C^*-algebra A is said to be amenable if every derivation from A into X^* is inner for all Banach A-modules X [6, p. 60].

For A a C^*-algebra, let $A \hat{\otimes} A$ be the completion of the algebraic tensor product $A \otimes A$ in the greatest cross-norm. We can identify $(A \hat{\otimes} A)^*$ with the space of bounded bilinear functionals on $A \times A$. We can make $A \hat{\otimes} A$, and hence $(A \hat{\otimes} A)^*$, into Banach A-modules by defining

$$a(b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c)a = b \otimes ca;$$

or by defining

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$$a \circ (b \otimes c) = b \otimes ac \quad \text{and} \quad (b \otimes c) \circ a = ba \otimes c.$$

The representations corresponding to these module actions commute. For example,

$$a \circ (b(c \otimes d)) = b(a \circ (c \otimes d)), \qquad a \circ ((c \otimes d)b) = (a \circ (c \otimes d))b.$$

For A a C^*-algebra let $U(A)$ denote the set of unitaries in A. The following proposition was stated in [1], but the proof given there was garbled (the proof refers to an earlier theorem that was changed in revision). We take this opportunity to clarify the proof.

PROPOSITION 1. *Let A be a C^*-algebra with identity. Then the following three statements are equivalent*:

(a) *A is amenable.*

(b) *There is a bounded linear map T of $(A \hat{\otimes} A)^*$ into $C = \{f \in (A \hat{\otimes} A)^*: af = fa$ for all $a \in A\}$ such that T restricted to C is the identity on C and $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$ for all $a \in A$, $f \in (A \hat{\otimes} A)^*$.*

(c) *Let Y be a Banach A-module and X a two-sided A-submodule of Y. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X$, $u \in U(A)$. Then there is an $h \in Y^*$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y$, all $u \in U(A)$.*

PROOF. The implications (b) implies (c) and (b) implies (a) were proved in [1]. We prove that (a) implies (b): Let $Y = (A \hat{\otimes} A)^* \hat{\otimes} (A \hat{\otimes} A)$ be made into a Banach A-module by the operations $(f \otimes t)a = f \otimes ta$; $a(f \otimes t) = f \otimes at$ for $a \in A$, $f \in (A \hat{\otimes} A)^*$ and $t \in A \hat{\otimes} A$. Let Z be the closed linear span of elements of the form

$$(a \circ f) \otimes t - f \otimes (t \circ a) \quad \text{and} \quad (f \circ a) \otimes t - f \otimes (a \circ t)$$

where $a \in A$, $t \in A \hat{\otimes} A$, and $f \in (A \hat{\otimes} A)^*$.

Let W be the linear span of elements of the form $f \otimes t$, where $f \in C$ and $t \in A \hat{\otimes} A$. Then W is clearly an A-submodule of Y and a computation shows that Z is an A-submodule of Y. Let X be the closed linear span of W and Z. Then X is an A-submodule of Y and X/Z is an A-submodule of Y/Z. Now finish the proof of (a) implies (b) as in [1, p. 570]. Finally, we prove that (c) implies (b): First note that if X is an A-submodule of Y and $f \in X^*$, then $f(uxu^*) = f(x)$ for all $x \in X$ and $u \in U(A)$ if and only if $f(ax) = f(xa)$ for all $x \in X$ and $a \in A$. Then let Y, Z, W and X be as in the proof of (a) implies (b). Let $F \in X^*$ be defined by $F(f \otimes t) = f(t)$. Then F is zero on Z and induces a map $F_1 \in (X/Z)^*$. If the bar denotes the coset in X/Z, we have that

$$F_1(u(f \otimes t)^- u^*) = F_1((f \otimes t)^-)$$

for all $u \in U(A)$ and elements $f \otimes t \in X$. Hence, by (c), there is a $G_1 \in (Y/Z)^*$ which extends F_1 and such that

$$G_1(u(f \otimes t)^- u^*) = G_1((f \otimes t)^-)$$

for all $u \in U(A)$ and elements $f \otimes t$ in Y. Now define T: $(A \hat{\otimes} A)^* \to C$ by

$$T(f)(t) = G_1((f \otimes t)^-)$$

for $f \in (A \hat{\otimes} A)^*$ and $t \in A \hat{\otimes} A$. The mapping T has the desired properties.

Let G be a discrete group. For $g \in G$ define a unitary operator $u_g \in B(l^2(G))$ by

$$u_g(F)(h) = F(g^{-1}h)$$

where $F \in l^2(G)$ and $h \in G$. Let $C_r^*(G)$ be the C^*-algebra generated by the u_g, $g \in G$, and let $W^*(G)$ be the weak closure of $C_r^*(G)$. Recall that a discrete group G is called amenable if there is an invariant mean on the bounded functions on G [5].

It follows from [6, p. 82] that if A is a C^*-algebra weakly dense in $W^*(G)$, then G is amenable if A is *strongly* amenable (see [6] for the definition of strongly amenable).

PROPOSITION 2. *Let A be a C^*-algebra with $C_r^*(G) \subseteq A \subseteq W^*(G)$. If A is an amenable C^*-algebra, then G is an amenable group. Conversely, if G is an amenable group then $C_r^*(G)$ is an amenable C^*-algebra.*

PROOF. Suppose that A is amenable. Then $B(l^2(G))$ is a Banach A-module and A is itself a two-sided A-submodule of $B(l^2(G))$. Let $\delta \in l^2(G)$ be the function on G which is one at the identity and zero elsewhere and define $f \in A^*$ by $f(a) = (a\delta,\delta)$. Then $f(ab) = f(ba)$ for all $a,b \in A$ [12, p. 164]. Then by part (c) of Proposition 1 there is an $h \in B(l^2(G))^*$ such that h extends f and $h(uyu^*) = h(y)$ for all $u \in U(A)$, $y \in B(l^2(G))$. Since $(h + h^*)/2$ will also have the same property, we may assume that h is a selfadjoint linear functional on $B(l^2(G))$. We now write $h = h^+ - h^-$ in its positive and negative parts [3, 12.3] and use an idea of Effros and Hahn [4, p. 25] to show that we can replace h by a state of $B(l^2(G))$. Indeed, for u a fixed element of $U(A)$ let g_1,g_2 in $B(l^2(G))^*$ be defined by

$$g_1(y) = h^+(uyu^*), \qquad g_2(y) = h^-(uyu^*),$$

where $y \in B(l^2(G))$. Then $h = g_1 - g_2$, $g_1 \geqslant 0$, $g_2 \geqslant 0$ and

$$\begin{aligned}\|h\| \leqslant \|g_1\| + \|g_2\| &= g_1(e) + g_2(e)\\ &= h^+(e) + h^-(e) = \|h^+\| + \|h^-\| = \|h\|,\end{aligned}$$

where e is the identity of $B(l^2(G))$. Hence by [3, 12.3.4] we have that $g_1 = h^+$, $g_2 = h^-$. Thus $h^+(uyu^*) = h^+(y)$ and $h^-(uyu^*) = h^-(y)$. Not both h^+ and h^- are identically zero, since h extends f, hence there exists a state h_1 on $B(l^2(G))$ such that $h_1(y) = h_1(u_g y u_g^*)$ for all $y \in B(l^2(G))$ and $g \in G$. For $\phi \in l^\infty(G)$ let $M_\phi \in B(l^2(G))$ be the operator which is multiplication by ϕ. Then for each $g \in G$ we have that $u_g M_\phi u_g^* = M_{\phi_g}$, where $\phi_g \in l^\infty(G)$ is defined by $\phi_g(h) = \phi(g^{-1}h)$. Now let ρ: $l^\infty(G) \to \mathbf{C}$ be defined by $\rho(\phi) = h_1(M_\phi)$. Then $\rho(1) = 1$, $\rho \geqslant 0$ and

$$\rho(\phi_g) = h_1(u_g M_\phi u_g^*) = h_1(M_\phi) = \rho(\phi).$$

Hence ρ is a left-invariant mean on $l^\infty(G)$ and G is amenable.

Conversely, suppose G is an amenable discrete group and let Y be a Banach $C_r^*(G)$-module with X a two-sided $C_r^*(G)$-submodule of Y. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $u \in U(C_r^*(G))$ and $x \in X$. Let $h \in Y^*$ be any extension of f and let $h_g(y) = h(u_g y u_g^*)$ for $g \in G, y \in Y$. Let ρ be an invariant mean on $l^\infty(G)$ and define $F \in Y^*$ by $F(y) = \rho(h_g(y))$. Then F extends f and $F(u_g y u_g^*) = F(y)$ for all $y \in Y$, $g \in G$. Thus $F(u_g y)$

$= F(yu_g)$ for all $y \in Y$, $g \in G$. The set $\{a \in C_r^*(G)\colon F(ay) = F(ya)$ for all $y \in Y\}$ is a Banach algebra which contains all u_g, hence $F(ay) = F(ya)$ for all $a \in C_r^*(G)$ and $F(uyu^*) = F(y)$ for all $u \in U(C_r^*(G))$. Thus $C_r^*(G)$ is amenable by Proposition 1.

We remark that C. Lance [7, Theorem 4.2] has proved that G is amenable if and only if $C_r^*(G)$ is nuclear. We do not know the relationship (if any) that exists between amenable and nuclear C^*-algebras. Proposition 2 shows that many nonamenable C^*-algebras exist. In fact if G is the free group on two generators and $C_r^*(G) \subseteq A \subseteq W^*(G)$, then A is not amenable. In particular, there exist nonamenable II_1-factors. We do not know if G amenable implies that $W^*(G)$ is amenable. S. Sakai has asked if the hyperfinite II_1-factor is amenable. We do not know the answer to this question.

3. **Finite operators and amenable algebras.** Let $A \in B(H)$, H a Hilbert space. The operator A is called finite if there is a state f on $B(H)$ such that $f(AB) = f(BA)$ for all $B \in B(H)$ [15].

PROPOSITION 3. *Let G be a discrete group and let $T \in B(l^2(G))$ be such that $C_r^*(G) \subseteq C^*(T) \subseteq W^*(G)$. Then T is a finite operator if and only if G is an amenable group.*

PROOF. If G is an amenable group then there exists a positive linear map F: $B(l^2(G)) \to W^*(G)$ such that $F(BA) = F(B)A$ and $F(AB) = AF(B)$ for all $B \in B(l^2(G))$ and $A \in W^*(G)$ [11, 4.4.21]. Let $f \in (W^*(G))^*$ be defined by $f(A) = (A\delta,\delta)$ where $\delta \in l^2(G)$ is the function which is one at the identity and zero elsewhere. Then

$$f(F(TB)) = f(TF(B)) = f(F(B)T) = f(F(BT))$$

for all $B \in B(l^2(G))$. Hence T is a finite operator. Conversely, assume T is a finite operator and suppose f is a state on $B(l^2(G))$ such that $f(AT) = f(TA)$ for all $A \in B(l^2(G))$. It then follows that $f(AB) = f(BA)$ for all $A \in B(l^2(G))$ and $B \in C^*(T)$. In particular, $f(u_gAu_g^*) = f(A)$ for all $A \in B(l^2(G))$ and $g \in G$. Let $M_\phi \in B(l^2(G))$ be multiplication by $\phi \in l^\infty(G)$ and define ρ on $l^\infty(G)$ by $\rho(\phi) = f(M_\phi)$. Then since $M_{\phi_g} = u_gM_\phi u_g^*$, we have that $\rho(\phi_g) = \rho(\phi)$ for all $g \in G$ and ρ is a left-invariant mean on $l^\infty(G)$; thus G is amenable.

It is known that if $T \in B(H)$ is a finite operator, then there is a representation of $C^*(T)$ whose weak closure is a finite factor [15]. However, Proposition 3 shows that the converse is not true.

COROLLARY 4. *There exists a nonfinite operator T which generates a type II_1-factor.*

PROOF. Let G be the free group on two generators c and d. We proceed as in [10, p. 453]. Let $u_c = A_c + iB_c$ and $u_d = A_d + iB_d$ be the Hermitian decompositions of u_c and u_d. There exist countable families of projections $E_{c,n}$, $F_{c,n} \in \{u_c\}''$ such that $A_c \in C^*(E_{c,n}\colon 1 \leqslant n) =$ the C^*-algebra generated by the $E_{c,n}$ and the identity, and $B_c \in C^*(F_{c,n}\colon 1 \leqslant n)$. Then $C^*(E_{c,n}, F_{c,n}\colon n \geqslant 1)$ is abelian and is generated as a Banach algebra by a countable family of idempotents; hence by Rickart's Lemma $C^*(E_{c,n}, F_{c,n}\colon n \geqslant 1) = C^*(H_c)$ for some self-adjoint operator H_c [14, p. 67]. Then $u_c \in C^*(H_c) \subseteq \{u_c\}''$.

Likewise there is a selfadjoint operator H_d such that $u_d \in C^*(H_d) \subseteq \{u_d\}''$. Let $H = H_c + iH_d$. Then $H \in W^*(G)$ and u_c, $u_d \in C^*(H)$. Then $C_r^*(G) \subseteq C^*(H) \subseteq W^*(G)$ and H generates a type II_1-factor, but H is not a finite operator by Proposition 3.

We remark that there exist amenable groups G such that $C_r^*(G) \subseteq C^*(T) \subseteq W^*(T)$ for some operator T. In fact if G is the "rational $ax + b$" group then such a T can be constructed by the same method as that used in Corollary 4.

4. **Finite operators.** The following proposition was conjectured by J. P. Williams [16, p. 279]. The proof uses the idea of Effros and Hahn [4, p. 25] that was used in the proof of Proposition 2.

PROPOSITION 5. *Let $A \in B(H)$ be such that there is a nonzero selfadjoint linear functional f on $B(H)$ such that $f(AB) = f(BA)$ for all $B \in B(H)$. Then A is a finite operator.*

PROOF. Let $C = \{T \in B(H): f(TB) = f(BT)$ for all $B \in B(H)\}$. Then it is easy to see that C is a C^*-subalgebra of $B(H)$ which contains A, hence $C^*(A) \subseteq C$. Let $f = f^+ - f^-$. Then as in the proof of Proposition 2 it follows that $f^+(UBU^*) = f^+(B)$ and $f^-(UBU^*) = f^-(B)$ for all $U \in U(C)$ and $B \in B(H)$. Thus $f^+(TB) = f^+(BT)$ and $f^-(TB) = f^-(BT)$ for all $B \in B(H)$ and $T \in C$, hence $f^+(AB) = f^+(BA)$ and $f^-(AB) = f^-(BA)$ for all $B \in B(H)$. Since at least one of f^+, f^- must be nonzero, it follows that A must be a finite operator.

For $\mathbb{S}_1, \mathbb{S}_2$ two subsets of $B(H)$ let $[\mathbb{S}_1, \mathbb{S}_2]$ denote the linear span of the commutators $S_1S_2 - S_2S_1$ where $S_i \in \mathbb{S}_i$, $i = 1,2$. For $A \in B(H)$ let δ_A be the inner derivation of $B(H)$ induced by A, $\delta_A(B) = BA - AB$, and let $R(\delta_A)$ be the range of δ_A.

COROLLARY 6. *For an operator $A \in B(H)$ the following are equivalent*:

(1) *A is finite,*

(2) *$[C^*(A), B(H)]$ is not norm dense in $B(H)$,*

(3) *the linear span of $R(\delta_A) \cup R(\delta_{A^*})$ is not norm dense in $B(H)$,*

(4) *the set of finite sums $\Sigma(X_i - U_iX_iU_i^*)$ where each $X_i \in B(H)$ and $U_i \in U(C^*(A))$ is not norm dense in $B(H)$.*

PROOF. The proof is immediate from Proposition 5 since the sets in (2), (3) and (4) are $*$-stable sets and are hence not norm dense if and only if there is a nonzero selfadjoint continuous linear functional which is zero on the set in question.

The condition (2) was conjectured by Williams in [16]. Condition (3) should be contrasted with Stampfli's result that $R(\delta_A)$ is never norm dense for any A [13].

We now denote by $\mathrm{Fin}(H)$ the set of finite operators in $B(H)$. The following two propositions concern representations of finite operators.

PROPOSITION 7. *Suppose $T \in B(H)$ and π: $C^*(T) \to B(H_\pi)$ is a $*$-representation such that $\pi(T) \in \mathrm{Fin}(H_\pi)$. Then $T \in \mathrm{Fin}(H)$.*

PROOF. Let π_0: $B(H) \to B(K)$ be the "extension" representation of π, where H_π is a subspace of K and $\pi_0(A)|H_\pi = \pi(A)$ for all $A \in C^*(T)$ [3, 2.10.2]. Let P be the projection of K onto H_π. Let g be a state of $B(H_\pi)$ such that $g(\pi(T)S) = g(S\pi(T))$ for all $S \in B(H_\pi)$, and define h on $B(H)$ by $h(X) = g(P\pi_0(X)P|H_\pi)$. Then h is a state on $B(H)$ and since H_π reduces $\pi_0(T)$ it is easily seen that $h(XT) = h(TX)$ for all $x \in B(H)$ and $T \in \operatorname{Fin}(H)$.

For each positive integer n let $R_n(H)$ denote the set of operators on H that have an n-dimensional reducing subspace. It is shown in [15] that $R_n(H)^-$ is contained in $\operatorname{Fin}(H)$. Whether or not $\bigcup_n R_n(H)$ is dense in $\operatorname{Fin}(H)$ is an open question. The following proposition is the analogue of Proposition 7 for the set $(\bigcup_n R_n(H))^-$. We do not know if the irreducibility assumption in this proposition can be omitted.

PROPOSITION 8. *Let $T \in B(H)$ and suppose π: $C^*(T) \to B(H_\pi)$ is an irreducible $*$-representation such that $\pi(T)$ is in the norm closure of $\bigcup_n R_n(H_\pi)$. Then T is in the norm closure of $\bigcup_n R_n(H)$.*

PROOF. We may assume that H_π is a subspace of H. Let $\varepsilon > 0$ be given and choose $A \in B(H_\pi)$ such that $\|\pi(T) - A\| < \varepsilon$ and A has a reducing subspace H_0 of finite dimension n. Let E_0 be the projection of H onto H_0 and let $x_1, x_2, \ldots, x_n$ be an orthonormal basis for H_0. Then by [2, Lemma, p. 342] there exists a unitary $U \in B(H)$ such that

$$\|UBU^*x_i - \pi(B)x_i\| \leqslant \varepsilon/n^{\frac{1}{2}}$$

for each i, $1 \leqslant i \leqslant n$, and $B = T$ or T^*. Then $\|(UBU^* - \pi(B))E_0\| \leqslant \varepsilon$ for $B = T$ or T^* and we have

$$\begin{aligned}\|(UTU^* - E_0AE_0)E_0\| &= \|(UTU^* - A)E_0 + (AE_0 - E_0AE_0)\| \\ &= \|(UTU^* - A)E_0\| \\ &\leqslant \|(UTU^* - \pi(T))E_0\| + \|(\pi(T) - A)E_0\| \leqslant 2\varepsilon,\end{aligned}$$

where we have used the fact that H_0 reduces A. The same inequality also holds for T^* and A^* in place of T and A. Hence

$$\begin{aligned}&\|UTU^* - (E_0AE_0|H_0 \oplus E_0^\perp UTU^*|E_0^\perp)\| \\ &\qquad \leqslant \|(UTU^* - (E_0AE_0|H_0 \oplus E_0^\perp UTU^*|E_0^\perp))E_0\| \\ &\qquad\quad + \|E_0(UTU^* - (E_0AE_0|H_0 \oplus E_0^\perp UTU^*|E_0^\perp))E_0^\perp\| \\ &\qquad \leqslant 2\varepsilon + 2\varepsilon.\end{aligned}$$

Thus T is in the norm closure of $(\bigcup_n R_n(H))^-$.

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