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NOTIONS OF SPANNING SURFACE EQUIVALENCE

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ABSTRACT. We show that two natural notions of spanning surface equivalence differ for minimal spanning surfaces of knots in S^3.

1. **Introduction.** Every (tame) knot in S^3 is spanned by a (tame) orientable surface [8]; an orientable spanning surface of smallest possible genus is called a *minimal spanning surface*. Spanning surfaces F and F' of a knot K in S^3 are *weakly equivalent* if there is an autohomeomorphism of S^3 taking F to F' and preserving the orientations of both S^3 and K. The surfaces F and F' are *strongly equivalent* if there is an isotopic deformation of S^3, fixing K (throughout the isotopy), and taking F to F'. Alford, Daigle, Lyon, Schaufele, and Trotter have given examples of knots which have minimal spanning surfaces F and F' which are not weakly equivalent [1]–[4], [6], [10]. We show here that there are knots K possessing minimal spanning surfaces F and F' which are weakly equivalent but not strongly equivalent.

2. **The construction.** Write $S^3 = R^3 \cup \infty$, and set

$$B_1 = \{(y_1,y_2,y_3): y_1 \geqslant 0\} \cup \infty, \quad B_2 = \{(y_1,y_2,y_3): y_1 \leqslant 0\} \cup \infty,$$

$$S^2 = \{(y_1,y_2,y_3): y_1 = 0\} \cup \infty,$$

$$a = \{(y_1,y_2,y_3): y_1 = 0, -1 \leqslant y_2 \leqslant 1, y_3 = 0\}$$

and $x = (0,0,1)$. Let h be the orientation preserving autohomeomorphism of S^3 given by $h(y_1,y_2,y_3) = (-y_1,-y_2,y_3)$; note that $h(a) = a$ and that h reverses the orientation of a. By [1], [2], we can find a knot k possessing minimal spanning surfaces S and S' with $\pi_1(S^3 - S) \not\approx \pi_1(S^3 - S')$. We may assume that k, S, and S' lie in B_1 and that $k \cap S^2 = S \cap S^2 = S' \cap S^2 = a$. Let K be the composite knot $k \# h(k) = (k \cup h(k)) - \operatorname{int}(a)$, and set $F = S \cup h(S')$ and $F' = S' \cup h(S)$; F and F' are minimal spanning surfaces of K [5, p. 141]. Since $h^2 = \mathrm{id}$, $h(F) = h(S \cup h(S')) = h(S) \cup S' = F'$. Also, since h reverses the orientation of a, h preserves the orientation of K. Therefore, F and F' are weakly equivalent. We will show, however, that F and F' are not strongly equivalent.

3. **Distinguishing between F and F'.** We will prove that F and F' are not strongly equivalent by demonstrating that if they were, it would then follow

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that $\pi_1(S^3 - S, x) \simeq \pi_1(S^3 - S', x)$, which is, by hypothesis, false. The fact that $\pi_1(S^3 - S, x)$ would be isomorphic to $\pi_1(S^3 - S', x)$ if F and F' were strongly equivalent will follow from a careful examination of the spaces and groups involved in our construction, together with an application of the algebraic consequences of the existence of a strong equivalence.

Let u_1, u_2, o_1, and o_2 be inclusion maps in the following commutative diagram:

$$\begin{tikzcd} & (S^2 - K) \arrow[dl, "u_1"'] \arrow[dr, "u_2"] & \\ (B_1 - K) \arrow[dr, "o_1"'] & & (B_2 - K) \arrow[dl, "o_2"] \\ & (S^3 - K) & \end{tikzcd}$$

(We let $(A - B)$ denote the set of all points which are in A but not in B, even when B is not a subset of A. Thus, for example, $(S^2 - K) = (S^2$ minus two points).) Letting n and n_2 be the natural maps in the commutative diagram

$$\begin{tikzcd} \mathbf{Z} = \pi_1(S^2 - K, x) \arrow[r, "(u_2)_*"] \arrow[d, "n"] & \pi_1(B_2 - K, x) \arrow[d, "n_2"] \\ \mathbf{Z} = H_1(S^2 - K) \arrow[r, "(u_2)_{**}"] & H_1(B_2 - K) = \mathbf{Z} \end{tikzcd}$$

and observing that n and $(u_2)_{**}$ are isomorphisms, we see that the homomorphism $(u_2)_*$ has a left inverse $v = n^{-1} \circ ((u_2)_{**})^{-1} \circ n_2$, which abelianizes $\pi_1(B_2 - K, x)$. Letting ϕ_1 be the identity homomorphism of $\pi_1(B_1 - K, x)$ to itself and letting $\phi_2\colon \pi_1(B_2 - K, x) \to \pi_1(B_1 - K, x)$ be the homomorphism $(u_1)_* \circ v$, we see that

$$\phi_1 \circ (u_1)_* = (u_1)_* = (u_1)_* \circ v \circ (u_2)_* = \phi_2 \circ (u_2)_*,$$

so that, by the Seifert-Van Kampen theorem, there is a homomorphism $\phi\colon \pi_1(S^3 - K, x) \to \pi_1(B_1 - K, x)$ such that $\phi \circ (o_1)_* = \phi_1 = \mathrm{id}$ and $\phi \circ (o_2)_* = \phi_2 = (u_1)_* \circ v$. Since v abelianizes $\pi_1(B_2 - K, x)$, we see that $\phi \circ (o_2)_*$ kills the commutator subgroup of $\pi_1(B_2 - K, x)$.

Now consider the following commutative diagram, in which all arrows are given by inclusion:

$$\begin{tikzcd} (B_1 - S) \arrow[r, "e_1"] \arrow[d, "i_1"] & (S^3 - F) \arrow[d, "i"] & (B_2 - h(S')) \arrow[l, "e_2"'] \arrow[d, "i_2"] \\ (B_1 - K) \arrow[r, "o_1"] & (S^3 - K) & (B_2 - K) \arrow[l, "o_2"'] \end{tikzcd}$$

By the Seifert-Van Kampen theorem, $\phi \circ i_*(\pi_1(S^3 - F, x))$ is generated by $\phi \circ i_* \circ (e_1)_*(\pi_1(B_1 - S, x))$ and $\phi \circ i_* \circ (e_2)_*(\pi_1(B_2 - h(S'), x))$. Now

$$\phi \circ i_* \circ (e_1)_* = \phi \circ (o_1)_* \circ (i_1)_* = \mathrm{id} \circ (i_1)_* = (i_1)^*,$$

which, by Dehn's lemma and the loop theorem, is injective since S is minimal (cf. [7, pp. 27–32] or [9]). Consequently

$$\phi \circ i_* \circ (e_1)_*(\pi_1(B_1 - S, x)) \simeq \pi_1(B_1 - S, x) \simeq \pi_1(S^3 - S, x).$$

Also, $\phi \circ i_* \circ (e_2)_* = \phi \circ (o_2)_* \circ (i_2)_* = 0$, since the image of $(i_2)_*$ is in the commutator subgroup of $\pi_1(B_2 - K, x)$, which is killed by $\phi \circ (o_2)_*$. Therefore, $\phi \circ i_*(\pi_1(S^3 - F, x)) \simeq \pi_1(S^3 - S, x)$. Similarly, letting $i'\colon (S^3 - F') \to (S^3 - K)$ be the inclusion map, we have

$$\phi \circ (i')_*(\pi_1(S^3 - F', x)) \simeq \pi_1(S^3 - S', x).$$

THEOREM. *F and F' are not strongly equivalent.*

PROOF. Suppose the contrary. Then there is an isotopic deformation $J\colon S^3 \times I \to S^3$ such that, for each t, $J_t(K) = K$, and $J_1(F) = F'$. Since $x \in (S^3 - (F \cup F'))$, we may assume as well that $J_1(x) = x$. Then $(J_1 | S^3 - K)_*$ is the inner automorphism of $\pi_1(S^3 - K, x)$ given by conjugation by the element of $\pi_1(S^3 - K, x)$ represented by the path of x during the isotopy J. We see also that

$$(J_1 | S^3 - K)_*(u_*(\pi_1(S^3 - F, x))) = (i')_*(\pi_1(S^3 - F', x)),$$

since

$$J_1(S^3 - F) = (S^3 - F').$$

Therefore, $i_*(\pi_1(S^3 - F, x))$ and $(i')_*(\pi_1(S^3 - F', x))$ are conjugate subgroups of $\pi_1(S^3 - K, x)$. Consequently

$$\phi \circ i_*(\pi_1(S^3 - F, x)) \quad \text{and} \quad \phi \circ (i')_*(\pi_1(S^3 - F', x))$$

are conjugate subgroups of $\pi_1(B_1 - K, x)$; in particular,

$$\phi \circ i_*(\pi_1(S^3 - F, x)) \simeq \phi \circ (i')_*(\pi_1(S^3 - F', x)),$$

or $\pi_1(S^3 - S, x) \simeq \pi_1(S^3 - S', x)$, a contradiction. ||

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