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HOMEOMORPHISMS OF $B^k \times T^n$

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ABSTRACT. An elementary proof is given that a homeomorphism of $B^k \times T^n$ rel ∂ that is homotopic to the identity is pseudoisotopic to the identity $\forall k, n$. The key step is to show that each homeomorphism is pseudoisotopic to any standard finite cover.

Let $A_0^{\mathrm{Top}}(B^k \times T^n)$ denote the pseudoisotopy classes of (self-) homeomorphisms of $B^k \times T^n$ rel ∂ which are homotopic rel ∂ to the identity. We will show $A_0^{\mathrm{Top}}(B^k \times T^n) = 0\ \forall k, n$. This should be contrasted with the corresponding PL result, where $A_0^{\mathrm{PL}}(B^k \times T^n) = H^{2-k}(T^n; \mathbf{Z}_2)$ for $n + k \geqslant 5$ (cf. [2]); in particular, $A_0^{\mathrm{PL}}(B^2 \times T^3) = \mathbf{Z}_2$, and thus $\mathbf{Z}_2$ is responsible for the failure of the Hauptvermutung. That $A_0^{\mathrm{Top}}(B^k \times T^n) = 0$ for $k + n \geqslant 5$ was known [6] using topological surgery. However, our result is completely elementary, relying only on the results of Lawson [3], the local contractibility of the homeomorphism group of a compact manifold [1], and an easy inductive argument.

We review the main result of [3], rephrased in terms relevant for our application. It says that there is an exact sequence.

$$0 \to A_0^{\mathrm{Top}}(B^{k+1} \times T^{n-1}) \xrightarrow{i} A_0^{\mathrm{Top}}(B^k \times T^n)$$
$$\xrightarrow{j} IC_0(B^k \times T^{n-1})/SC(B^k \times T^{n-1}) \to 0.$$

The map i comes from regarding $B^k \times T^n$ as $B^k \times T^{n-1} \times I/B^k \times T^{n-1} \times \{0, 1\}$. The map j takes a homeomorphism g of $B^k \times T^n$, regards it as a homeomorphism of $B^k \times T^{n-1} \times S^1$, lifts it to $\bar{g}$: $B^k \times T^{n-1} \times \mathbf{R}$ so that $\bar{g}(B^k \times T^{n-1} \times 0)$ lies above $B^k \times T^{n-1} \times 0$, and then $j([g])$ is the invertible $B^k \times T^{n-1}$-cobordism $(W, i_0, \bar{g}i_0, I)$, where W is the region between $B^k \times T^{n-1} \times 0$ and $\bar{g}(B^k \times T^{n-1} \times 0)$,

$$i_0\colon B^k \times T^{n-1} \to B^k \times T^{n-1} \times 0 \subset B^k \times T^{n-1} \times \mathbf{R}$$

is the inclusion and I is a trivialization along the boundary (cf. [3], [4]). The main property of j which we need is that if $\tilde{g}$ denotes the d-fold cover of g making the following diagram

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$$\begin{array}{ccc} B^k \times T^{n-1} \times \mathbf{R} & \xrightarrow{\bar{g}} & B^k \times T^{n-1} \times \mathbf{R} \\ \downarrow & & \downarrow \\ B^k \times T^{n-1} \times S^1 & \xrightarrow{\tilde{g}} & B^k \times T^{n-1} \times S^1 \\ \downarrow & & \downarrow \\ B^k \times T^{n-1} \times S^1 & \xrightarrow{g} & B^k \times T^{n-1} \times S^1 \end{array}$$

commute, then $j([\tilde{g}]) = j([g])$.

THEOREM. $A_0^{\mathrm{Top}}(B^k \times T^n) = 0$.

PROOF. We proceed by induction on n. If $n = 0$ then the result holds by the Alexander isotopy. Our inductive hypothesis then simplifies the exact sequence to an isomorphism

$$A_0^{\mathrm{Top}}(B^k \times T^n) \simeq IC_0(B^k \times T^{n-1})/SC(B^k \times T^{n-1}).$$

Now if g represents an element of $A_0^{\mathrm{Top}}(B^k \times T^n)$, and $\tilde{g}$ is a d-fold cover as above, we have $j([\tilde{g}]) = j([g])$, hence $[\tilde{g}] = [g]$. By interchanging the factors of S^1 in T^n, this argument then implies that if $\tilde{g}$ is any standard finite cover of g, then $[\tilde{g}] = [g]$. But by [1] and [6, Lemma 4.1], if $\tilde{g}$ is the standard d^n-fold cover for d large enough, $\tilde{g}$ is isotopic (hence pseudoisotopic) to the identity.

REMARK. The reader may find further applications of the exact sequence of [3], including others of relevance to the failure of the Hauptvermutung, in [3], [4], and [5].

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