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MONIC AND MONIC FREE IDEALS IN A POLYNOMIAL SEMIRING

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ABSTRACT. Two classes of ideals are introduced in a polynomial semiring $S[x]$, where S is a commutative semiring with an identity. A structure theorem is given for each class.

1. **Introduction.** While much is known about ideals in rings and polynomial rings, little is known about ideals in polynomial semirings. In this paper, two classes of ideals in a polynomial semiring will be explored and a structure theorem for each class will be presented.

2. **Fundamentals.** There are different definitions of a semiring appearing in the literature. However, the definition used in [1] will be used throughout this paper. This definition is given as follows:

2.1. DEFINITION. A set S together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ will be called a semiring provided $(S, +)$ is an abelian semigroup with a zero, $(S, \cdot)$ is a semigroup, and multiplication distributes over addition from the left and from the right.

A semiring S is said to be commutative if $(S, \cdot)$ is a commutative semigroup. A semiring S is said to have an identity if there exists $1 \in S$ such that $1 \cdot x = x \cdot 1$ for each $x \in S$.

2.2. DEFINITION. A semiring S is said to be a strict semiring if $a \in S$, $b \in S$ and $a + b = 0$ imply $a = b = 0$.

The set of nonnegative integers under the usual operations of addition and multiplication is a strict semiring.

2.3. DEFINITION. A subset I of a semiring S will be called an ideal in S if I is an additive subsemigroup of $(S, +)$, $IS \subset I$ and $SI \subset I$.

2.4. DEFINITION. An ideal I in a semiring S will be called a k-ideal if $a \in I$, $b \in S$ and $a + b \in I$ imply $b \in I$.

2.5. DEFINITION. An ideal M in $S[x]$, where S is a commutative semiring with an identity, will be called monic if $\sum a_i x^i \in M$ implies $a_i x^i \in M$ for each $i \in \{0, 1, 2, \ldots, n\}$.

2.6. DEFINITION. An ideal F in $S[x]$, where S is a commutative semiring with an identity, will be called monic free if M is a monic ideal such that $M \subset F$

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then $M = \{0\}$. Ideals which are neither monic nor monic free will be called mixed.

2.7. EXAMPLES. Let Z be the integers and $Z[x]$ the ring of polynomials over Z. Let $ax^i \in Z[x]$ where $i > 0$ and $a \neq 0$. Then, clearly (ax^i) is a monic ideal in $Z[x]$. Now let $F = (x + 1)$ be the ideal in $Z[x]$ generated by $x + 1$. Then F is a monic free ideal in $Z[x]$. To see this, let M be a monic ideal such that $M \subset F$. If $M \neq 0$, then M being monic assures the existence of an element $ax^i \in M$ such that $a \neq 0$ and $i > 0$. Without loss of generality, we may assume that i is even. Thus, $ax^i \in F$, whence $ax^i = g(x)(x + 1)$ for some polynomial $g(x)$ in $Z[x]$. Replacing x by -1 yields $a = 0$, a contradiction. Hence $M = 0$ and F is monic free.

Throughout this paper, unless otherwise stated, S will be a commutative semiring with an identity and $S[x]$ will be the semiring of polynomials over S in the indeterminate x.

3. **Monic ideals.** Let $\{I_n\}$ be an ascending chain of ideals in a semiring S and $I^* = \{\sum a_i x^i \in S[x] \| a_i \in I_i\}$.

3.1. THEOREM. *I^* is a monic ideal in $S[x]$.*

PROOF. Let $f = a_n x^n + \cdots + a_0 \in I^*$, $g = b_m x^m + \cdots + b_0 \in I^*$, and $h = c_k x^k + \cdots + c_0 \in S[x]$. It follows from $(a_i + b_i) \in I_i$ that $f + g \in I^*$. Consider the product $hf = \sum p_t x^t$, where $p_t = \sum c_i a_j$ for $i + j = t$. Clearly $j \leqslant t$ and consequently, $a_j \in I_t$ since $\{I_n\}$ is an ascending chain. Hence $p_t = \sum c_i a_j \in I_t$ and $hf \in I^*$. Since $f \in I^*$, it follows that $a_i \in I_i$ and consequently $a_i x^i \in I^*$.

At this point the following question may be asked: Does every monic ideal in $S[x]$ come from an ascending chain of ideals in S? To answer this question, a method of constructing an ascending chain of ideals in S from a given ideal in $S[x]$ is needed.

Let A be an ideal in $S[x]$ and $A_i = \{a \in S |$ there is an $f \in A$ such that ax^i is a term of $f\}$.

3.2. THEOREM. *If A is an ideal in $S[x]$, then $\{A_n\}$ is an ascending chain of ideals in S.*

PROOF. For $a \in A_i$ and $b \in A_i$ there are polynomials $f \in A$ and $g \in A$ such that ax^i and bx^i are terms of f and g respectively. Consequently, $f + g \in A$, $(a + b)x^i$ is a term of $f + g$ and $a + b \in A_i$. If $c \in S$, then $cf \in A$ and it follows that cax^i is a term of cf. Consequently $ca \in A_i$. Since $xf \in A$, it is clear that ax^{i+1} is a term of xf and $A_i \subset A_{i+1}$.

For an ideal A in $S[x]$, the ascending chain of ideals $\{A_n\}$ in S will be called coefficient ideals. Let $A^* = \{\sum a_i x^i \in S[x] \| a_i \in A_i\}$. Then $A \subset A^*$ and Theorem 3.1 assures that A^* is a monic ideal in $S[x]$.

3.3. THEOREM. *An ideal A in $S[x]$ is monic if and only if $A = A^*$.*

PROOF. Theorem 3.1 assures that A is monic if $A = A^*$. Suppose A is monic and $f = a_n x^n + \cdots + a_0 \in A^*$. Then $a_i \in A_i$ and there is a polynomial $g_i \in A$ such that ax^i is a term of g_i. A being monic assures that $a_i x^i \in A$.

Consequently $f \in A$ and $A^* \subset A$. From $A \subset A^*$ it follows that $A = A^*$.

This theorem assures that all monic ideals in $S[x]$ can be structured from ascending chains of ideals in S. What about monic k-ideals?

3.4. THEOREM. *A monic ideal M in $S[x]$ is a k-ideal if and only if M_i is a k-ideal in S for each i.*

PROOF. Suppose M is a monic k-ideal in $S[x]$. Then $M = M^*$. Let $a \in M_i$, $b \in S$ and $a + b \in M_i$. Then $ax^i \in M$, $(a + b)x^i = ax^i + bx^i \in M$ and $bx^i \in M$ since M is a k-ideal. Consequently $b \in M_i$ and M_i is a k-ideal. Conversely, suppose each M_i is a k-ideal, $f = a_n x^n + \cdots + a_0 \in M$, $g = b_m x^m + \cdots + b_0 \in S[x]$ and $f + g \in M$. Then $a_i \in M_i$, $a_i + b_i \in M_i$ and it follows that $b_i \in M_i$. Consequently $b_i x^i \in M$ and $g \in M$. Therefore M is a k-ideal.

3.5. DEFINITION. For an ideal A in $S[x]$, the set $\overline{A} = \bigcap\{M \mid M$ is monic and $A \subset M\}$ will be called the monic closure of A.

It is evident that for any ideal A in $S[x]$, $\overline{A}$ is a monic ideal and consequently $\overline{A} = A^*$. It is easy to show that if $A \subset B$, then $\overline{A} \subset \overline{B}$.

3.6. THEOREM. *If A and B are ideals in $S[x]$, then $\overline{A} \cap \overline{B} = \overline{A \cap B}$.*

PROOF. $\overline{A} \cap \overline{B} \subset \overline{A \cap B}$ is clear. Let $f = \sum a_i x^i \in \overline{A \cap B}$. Since $\overline{A \cap B}$ is monic, $a_i x^i \in \overline{A \cap B}$. Thus there is a polynomial $g_i \in A \cap B$ such that $a_i x^i$ is a term of g_i. Now $g_i \in A$ and $g_i \in B$. Hence $a_i x^i \in \overline{A}$ and $a_i x^i \in \overline{B}$ and it follows that $a_i x^i \in \overline{A} \cap \overline{B}$. Consequently $f = \sum a_i x^i \in \overline{A} \cap \overline{B}$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Whence $\overline{A} \cap \overline{B} = \overline{A \cap B}$.

4. **Monic free ideals.** Of the monic free ideals in a polynomial semiring, the monic free k-ideals are the most interesting. The following lemma is important to the study of monic free k-ideals.

4.1. LEMMA. *If A is a k-ideal in $S[x]$, $f = a_n x^n + \cdots + a_0 \in A$, and τ is a nonnegative integer, then*

$$(a_n x^n + \cdots + a_{i+1} x^{i+1} + a_{i-1} x^{i-1} + \cdots + a_0)^{2\tau+1} + (a_i x^i)^{2\tau+1} \in A.$$

PROOF. By induction on τ. Let $f = h + g$ where $h = a_n x^n + \cdots + a_{i+1} x^{i+1} + a_{i-1} x^{i-1} + \cdots + a_0$ and $g = a_i x^i$. Assume that $h^{2\tau+1} + g^{2\tau+1} \in A$. Since A is an ideal it is clear that $[h^{2\tau+2} + g^{2\tau+2}]f \in A$ and $[h^{2\tau+1} + g^{2\tau+1}]hg \in A$. Then

$$\begin{aligned}[h^{2\tau+2} + g^{2\tau+2}]f &= [h^{2\tau+2} + g^{2\tau+2}](g + h) \\ &= h^{2\tau+3} + g^{2\tau+3} + [h^{2\tau+1} + g^{2\tau+1}]hg.\end{aligned}$$

Consequently $h^{2\tau+3} + g^{2\tau+3} \in A$, since A is a k-ideal, and $h^{2\tau+1} + g^{2\tau+1} \in A$ for all nonnegative integers τ.

4.2. THEOREM. *Let S be a strict semiring. A monic free ideal F in $S[x]$ with a finite basis is not a k-ideal.*

PROOF. Suppose that the theorem is false and F is a monic free k-ideal with a finite basis $B = \{g_1, g_2, \ldots, g_n\}$. Let $\Delta\colon S[x] \to Z^+$ be a function defined as follows: (1) if $h = b_n x^n + b_{n-p} x^{n-p} + \cdots + b_0$ has degree n and $b_{n-p} \neq 0$, then $\Delta(h) = p$, (2) if $h = ax^n$, $a \neq 0$, then $\Delta(h) = n$ and (3) if $h = 0$, then $\Delta(h) = 0$. Now let $\Delta(g_i) = c_i$. Since F is monic free, g_i contains at least two nonzero terms and consequently $c_i \geqslant 1$. Suppose $c = \max\{c_1, c_2, \ldots, c_n\}$ and consider $f = a_n x^n + a_{n-p} x^{n-p} + \cdots + a_0 \in F$. Clearly $\Delta(f) = p$ and it follows from Lemma 4.1 that $f_\tau = (a_n x^n)^{2\tau+1} + (a_{n-p} x^{n-p} + \cdots + a_0)^{2\tau+1} \in F$ for each nonnegative integer τ. Since p is fixed and $\Delta(f_\tau) = (2\tau + 1)p$, the sequence $\{(2\tau + 1)p\}$ is an increasing sequence of integers. Consequently, there is a λ such that $\Delta(f_\lambda) = (2\lambda + 1)p > c$. Also $f_\lambda \in F$ and B a basis for F assures that

$$(1)\quad f_\lambda = (a_n x^n)^{2\lambda+1} + (a_{n-p} x^{n-p} + \cdots + a_0)^{2\lambda+1} = h_1 g_1 + \cdots + h_n g_n$$

for $h_i \in S[x]$. At least one of the products, say $h_i g_i$ must produce a term of degree $(2\lambda + 1)n$, since $(a_n x^n)^{2\lambda+1}$ appears on the left side of (1). From $\Delta(g_i) = c_i$ it follows that $g_i = b_m x^m + b_{m-c_i} x^{m-c_i} + \cdots + b_0$. Moreover, h_i must have a term of the form $dx^{(2\lambda+1)n-m}$ and $dx^{(2\lambda+1)n-m} g_i = db_m x^{(2\lambda+1)n} + db_{m-c_i} x^{(2\lambda+1)n-c_i} + \cdots + db_0 x^{(2\lambda+1)n-m}$ is part of the product $h_i g_i$. Since S is a strict semiring, none of the terms in any of these products can vanish. Consequently the right side of (1) contains a term of degree $(2\lambda + 1)n - c_i$. A term of this degree is guaranteed because g_i must contain at least two nonzero terms. Since $(2\lambda + 1)p > c \geqslant c_i$, it follows that

$$(2)\quad \begin{aligned}(2\lambda + 1)n &> (2\lambda + 1)n - c_i \geqslant (2\lambda + 1)n - c > (2\lambda + 1)n - (2\lambda + 1)p \\ &= (2\lambda + 1)(n - p).\end{aligned}$$

The second highest term on the left side of (1) is $(2\lambda + 1)(n - p)$. Hence a term of degree $(2\lambda + 1)n - c_i$ cannot appear on the left side of (1) because of (2), a contradiction.

4.3. COROLLARY. *Let S be a strict semiring. If F is a monic free k-ideal in $S[x]$, then every basis for F is infinite.*

The above results make it possible to prove the following structure theorem for monic free k-ideals in $S[x]$, where S is a strict semiring.

4.4. THEOREM. *Let S be a strict semiring. If F is a monic free k-ideal in $S[x]$, then $F = \bigcup F_\alpha$ where $\{F_\alpha\}$ is a proper ascending chain of ideals.*

PROOF. Corollary 4.3 assures that F has an infinite basis, say, $B = \{g_\alpha\}$ for $\alpha \in A$. Well order the elements of B and let $F_0 = (g_0)$ and $F_\alpha = \sum_{\gamma<\alpha} F_\gamma + (g_\alpha)$. It is easy to see that $\{F_\alpha\}$ is a proper ascending chain of ideals and $F = \bigcup F_\alpha$.

4.5. EXAMPLE. Consider the integers Z and the nonnegative integers Z^+. Clearly Z is a semiring and Z^+ is a strict semiring. Define a mapping $\eta\colon Z^+[x] \to Z[i]$, $i = \sqrt{-1}$, by $\eta(f(x)) = f(i)$. It is clear that η is a semiring homomorphism and that $F = \ker \eta$ is a k-ideal in Z^+. Let M be a monic ideal such that $M \subset F$. If $ax^t \in M$, then $ax^t \in F$ and $\eta(ax^t) = ai^t = 0$. Hence $a = 0$ and it follows that $ax^t = 0$. Consequently $M = 0$ and F is monic free.

Now let $A_0 = (x^2+1)$, $A_1 = (x^6+1) + A_0, \ldots, A_n = (x^{4n+2}+1) + A_{n-1}, \ldots$, and let $A = \cup A_i$. Clearly $\{A_i\}$ is a proper ascending chain of ideals in $Z^+[x]$. If $f(x) \in A$, then there exists p such that $f(x) \in A_p$ and it follows that $f(x) = (x^2+1)f_0(x) + \cdots + (x^{4p+2}+1)f_p(x)$. Further,

$$\begin{aligned}\eta(f(x)) = f(i) &= (i^2+1)f_0(i) + \cdots + (i^{4p+2}+1)f_p(i)\\ &= 0\cdot f_0(i) + \cdots + 0\cdot f_p(i) = 0,\end{aligned}$$

and it follows that $f(x) \in F$. Consequently, $A \subset F$. Now suppose $f(x) = a_n x^n + \cdots + a_0 \in F$. Then $\eta(f(x)) = f(i) = 0$. Write $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ has only odd degree terms and $f_2(x)$ has only even degree terms. It follows from $f(i) = 0$ that $f_1(i) = 0$ and $f_2(i) = 0$. Using this and the fact that the coefficients of $f(x)$ are nonnegative integers, it is straightforward to show, by rearranging terms and factoring, that $f(x) = (x^2+1)g_0(x) + \cdots + (x^{4t+1}+1)g_t(x)$, where $g_i(x) \in Z^+[x]$. Hence $f(x) \in A$ and $F \subset A$. Consequently $A = F$. Thus F is a monic free k-ideal with an infinite basis.

Next, let $N = (x+1)$ be the ideal in $Z^+[x]$ generated by $x+1$. It follows from Example 2.7 that N is monic free in $Z^+[x]$. Thus N is a monic free ideal with a finite basis. Now assume that N is a k-ideal. It is clear that $(x+1)^2(x+1) = (x+1)^3 \in N$ and $3x(x+1) \in N$. Consequently,

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1 = x^3 + 1 + 3x(x+1)$$

and $x^3 + 1 \in N$. But this gives $x^3 + 1 = g(x)(x+1)$, for some $g(x) \in Z^+[x]$, which is impossible. Thus N is not a k-ideal.

5. **Mixed ideals.** When E is a mixed ideal in $S[x]$ one can consider the monic part of E and the monic free part of E. While E may contain many monic ideals, it also contains a "largest" monic ideal.

5.1. DEFINITION. When E is a mixed ideal in $S[x]$ the set $E^0 = \sum\{M_\alpha \mid M_\alpha$ is monic and $M_\alpha \subset E\}$ will be called the monic interior of E.

Obviously E^0 is a monic ideal in $S[x]$ and it follows from the definition of E^0 that E^0 is the maximal monic ideal contained in E. Also since E is mixed, $E^0 \neq E$.

5.2. DEFINITION. When E is a mixed ideal in $S[x]$ the set $bE = E - E^0$ will be called the boundary of E.

5.3. THEOREM. *Let S be a strict semiring. If E is a mixed ideal in $S[x]$ then $E = E_1 \cup E_2$ where E_1 is the maximal monic ideal contained in E and E_2 is monic free.*

PROOF. Since $E_1 = E^0$, it only remains to show that E_2 is monic free. Let $E_2 = (bE)$ be the ideal generated by bE. To show that E_2 is monic free it is sufficient to show that the ideal (bE) contains no nonzero elements of the form ax^i. Observe that the boundary bE can contain no term of the form ax^i since (ax^i) would be a monic ideal contained in E and $ax^i \in (ax^i) \subset E^0$. This is impossible since $bE \cap E^0 = \varnothing$. Consequently, the ideal (bE), being in a polynomial semiring, can contain no term of the form ax^i since bE is a basis for (bE) and S is a strict semiring. Thus E_2 is monic free.

It is noted here that for a mixed ideal in $S[x]$ the ascending chain $E^0 \subset E \subset \bar{E}$ is always proper.

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