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K_i OF UPPER TRIANGULAR MATRIX RINGS

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ABSTRACT. Standard techniques are used to compute K_i ($i = 0, 1, 2$) of generalized triangular matrix rings.

For R an associative ring with unit, let T denote the ring of upper triangular 2 by 2 matrices over R. Quillen[2] has announced that $K_i(T) \approx K_i(R) \oplus K_i(R)$ for all $i \geqslant 0$. The purpose of this note is to give a proof of a generalization of this theorem for $i = 0, 1, 2$ using standard techniques. This generalization is probably true for the higher K's.

For R and S rings with unit and M an R-S-bimodule, let T denote the ring of all upper triangular matrices of the form

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}, \qquad r \in R, s \in S, m \in M,$$

with addition and multiplication defined in the obvious way. We will prove the following:

THEOREM 1. *For $i = 0, 1, 2$ the canonical map $K_i(T) \to K_i(R) \oplus K_i(S)$ is an isomorphism.*

An induction argument yields

COROLLARY 2. *Let T_n be the ring of upper triangular n by n matrices over the ring R. Then for $i = 0, 1, 2$, $K_i(T_n) \approx K_i(R)^n$.*

1. **The cases** $i = 0, 1$. Let J denote the ideal of T which consists of those matrices whose only nonzero entries lie in M. As $J^2 = 0$, J is contained in the Jacobson radical of T. Thus T is J-adically complete and the map $K_0(T) \to K_0(T/J)$ is an isomorphism [**B**, Proposition 1.3, p. 449]. This yields the result in case $i = 0$.

Since the map $T \to T/J$ splits and $T/J \approx R \oplus S$, the exact sequence for an ideal yields

$$(1) \qquad 1 \to K_i(T,J) \to K_i(T) \to K_i(R) \oplus K_i(S) \to 1.$$

According to Swan [**Sw**, Theorem 2.1] $K_1(T,J) \approx 1 + J/W(T,J)$ where $W(T,J)$ is the subgroup of T^* (the group of units of T) generated by all

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elements of the form $(1+jt)(1+tj)^{-1}$, $t \in T, j \in J$. But an arbitrary element of $1+J$, $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$, lies in $W(T,J)$ as is seen by taking $t = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $j = \left(\begin{smallmatrix}0 & m\\ 0 & 0\end{smallmatrix}\right)$. Hence $K_1(T,J)$ is trivial and the result follows from exact sequence (1).

It should be noted that Swan's theorem is not really necessary. Upon observing that the diagonal elements of a matrix congruent to 1 modulo J are units, the usual process of row and column reduction together with the equation

$$\operatorname{diag}(1+j,1) = E_{12}(j)E_{21}(t)E_{12}(-j)E_{21}(-t)$$

yields the result.

2. **The case** $i = 2$. Throughout this section unexplained notation will be that of **[D-Sl]**. Let $a, b \in R$ be such that $1 + ab \in R^*$. Note that $1 + ba \in R^*$. For each pair of indices $\alpha = ij$, let $-\alpha$ denote the reversed pair, ji, and define

$$H_\alpha(a,b) = x_{-\alpha}(-b(1+ab)^{-1})x_\alpha(a)x_{-\alpha}(b)x_\alpha(-a(1+ba)^{-1}).$$

Denote by $H(n,R)$ the subgroup of $\mathrm{St}(n,R)$ generated by all elements $H_\alpha(a,b)$. Note that $\varphi(H_\alpha(a,b)) = \operatorname{diag}(u_1,\dots,u_n)$, where

$$u_i = 1+ab, \qquad u_j = (1+ba)^{-1},$$

and $u_k = 1$ for $k \neq i, j$. According to **[D-S1**, 8(a), p. 248**]** it follows that, for $H \in H(n,R)$, $n \geqslant 3$, $\varphi(H) = \operatorname{diag}(v_1,\dots,v_n)$,

$$(2) \qquad {}^{H}x_{kl}(r) = x_{kl}(v_k r v_l^{-1}).$$

In particular, it follows that $K_2(n,R) \cap H(n,R)$ is central in $\mathrm{St}(n,R)$ for all $n \geqslant 3$.

Let J be any ideal contained in the Jacobson radical of the ring R and define $H(n,J)$ to be the subgroup of $H(n,R)$ generated by all elements $H_\alpha(a,b)$ where at least one of a, b lies in J. We define $K_2(n,J)$ to be the kernel of the map $K_2(n,R) \to K_2(n,R/J)$. The techniques of **[St]** and **[S-D]** applied to the case of an arbitrary ring R immediately yield the first part of the following theorem.

THEOREM 3. *Let J be an ideal contained in the Jacobson radical of the ring R. Then $K_2(n,J) \subset H(n,J)$ for all $n \geqslant 3$. Consequently the maps $K_2(n,J) \to K_2(n+1,J) \to K_2(J)$ are surjective for all $n \geqslant 2$.*

The proof of surjectivity and the proof of Theorem 1 depend on certain identities in $H(n,R)$ listed below. Throughout this section we write $\alpha = ij$, $\beta = jk$, and $\gamma = ki$ for distinct integers i, j, k.

LEMMA 4. *The following identities are valid in* $\mathrm{St}(n,R)$:
(i) *If $H \in H(n,R)$ is such that $\varphi(H) = \operatorname{diag}(v_1,\dots,v_n)$, then for $n \geqslant 3$,*

$${}^{H}H_\alpha(a,b) = H_\alpha(v_i a v_j^{-1}, v_j b v_i^{-1}).$$

(ii) *If $n \geqslant 2$, then*

$$H_\alpha(a,b) = H_{-\alpha}(-b,-a)^{-1}.$$

(iii) *If* $n \geqslant 3$, *then*

$$H_\alpha(a_1 + a_2, b) = H_\alpha(a_2, b(1 + a_1 b)^{-1}) H_\alpha(a_1, b)$$

and

$$H_\alpha(a, b_1 + b_2) = H_\alpha(a, b_1) H_\alpha(a(1 + b_1 a)^{-1}, b_2).$$

(iv) *If* $n \geqslant 3$, *then*

$$H_\alpha(a, bc)^{-1} H_\beta(b, ca)^{-1} H_\gamma(c, ab)^{-1} = 1.$$

(v) *If* $n \geqslant 3$, *then*

$$H_\alpha(a,b) = H_{-\alpha}(b, -a(1 + ba)^{-1})$$

and

$$H_\alpha(a,b) = H_\alpha(a(1 + ba)^{-1}, -b)^{-1}.$$

Identity (i) is immediate from equation (2). The proofs of (ii)–(iv) involve writing certain expressions in the unique LHU form of Stein as in [**St**, Lemma 2.7] or [**S-D**, Proposition 1.1]. This is accomplished with the aid of the formula

$$^{x_\alpha(a)}x_{-\alpha}(b) = x_{-\alpha}(b(1 + ab)^{-1}) H_\alpha(a,b) x_\alpha(-aba(1 + ba)^{-1})$$

when $1 + ab \in R^*$.

To prove (ii) put the left and right sides of

$$^{x_\alpha(a)}x_{-\alpha}(b) = x_{-\alpha}(b)(^{x_{-\alpha}(-b)}x_\alpha(-a))^{-1} x_\alpha(-a)$$

in the LHU form. The first part of (iii) is obtained similarly by considering

$$^{x_\alpha(a_1+a_2)}x_{-\alpha}(b) = {}^{x_\alpha(a_2)x_\alpha(a_1)}x_{-\alpha}(b).$$

Note that we must assume that $n \geqslant 3$ in order to apply (2) to put the right-hand side in the LHU form. The second part of (iii) is obtained from the first part by applying (ii), taking inverses and renaming.

Part (iv) is obtained by using the Philip Hall identity

$$^{y}[x,[y^{-1},z]]\ {}^{z}[y,[z^{-1},x]]\ {}^{x}[z,[x^{-1},y]] = 1$$

with $x = x_\alpha(a)$, $y = x_\beta(b)$, and $z = x_\gamma(c)$. The expression is then simplified using the Steinberg relations and (2) with the final step an application of (ii). For an outline of the computation see [**D-S2**, Proposition 1.1] (cf. [**Sw**, Lemma 7.7]).

To prove the first part of (v), note that, if $n \geqslant 3$, then

$$^{H_\alpha(a,b)}x_{-\alpha}(b(1 + ab)) = x_{-\alpha}(b(1 + ab)^{-1}),$$

and hence

$$\begin{aligned} H_\alpha(a,b) &= x_{-\alpha}(b(1+ab)^{-1})H_\alpha(a,b)x_{-\alpha}(-b(1+ab)) \\ &= H_{-\alpha}(b,-a(1+ba)^{-1}). \end{aligned}$$

The last equality follows from the definition of H_α after one simplifies the preceding expression. The second part of (v) follows from the first by an application of (ii).

LEMMA 5. *Let J be an ideal contained in the Jacobson radical of the ring R. Then for all $n \geqslant 2$*

(a) *the map $K_2(n,R) \cap H(n,R) \to K_2(n+1,R) \cap H(n+1,R)$ is surjective and*

(b) *the map $K_2(n,J) \cap H(n,J) \to K_2(n+1,J) \cap H(n+1,J)$ is surjective.*

By taking $c = 1$ and $k = 1$ in (iv) and applying (i) we obtain

$$H_{ij}(a,b) = H_{j1}(b,a)^{-1}H_{1i}(1,ab)^{-1} = H_{1j}(-a,-b)H_{1i}(1,ab)^{-1}.$$

Thus elements of the form $H_{1j}(a,b)$ generate $H(n+1,R)$. By applying (i) we see that any $z \in H(n+1,R)$ can be written as $z = z_2 \cdots z_{n+1}$ where z_j is a product of elements of the form $H_{1j}(a,b)^{\pm 1}$. Thus if $z \in K_2(n+1,R) \cap H(n+1,R)$, we have $\varphi(z) = 1$. Hence $\varphi(z_j) = \operatorname{diag}(u_1,\ldots,u_{n+1})$ where $u_i = 1$ for $2 \leqslant i \leqslant n+1$. If $j > 2$, then ${}^{z_j}w_{2j}(1) = w_{2j}(1)$ by equation (2) and hence $z_j = {}^{w_{2j}(1)}z_j$. Now by [**Mi**, Corollary 9.4] it follows that z_j, and hence z, is a product of elements of the form $H_{12}(a,b)^{\pm 1}$. These elements clearly lie in the image of $K_2(2,R) \cap H(2,R)$, proving the first assertion. The proof of the second is analogous. This also yields the second part of Theorem 3.

We now proceed to the proof of Theorem 1 in the case $i = 2$. For all $n \geqslant 3$, $K_2(n, \ )$ preserves finite products, and thus there is a short exact sequence as in (1). We thus need only show that $K_2(n,J)$ is trivial for all $n \geqslant 3$. For the rest of this section, let $n \geqslant 3$. By Theorem 3 we may assume that the elements of $K_2(n,J)$ are products of elements of the form $H_{12}(a,b)^{\pm 1}$ and $H_{21}(a,b)^{\pm 1}$ where one of a, b is in J. Note that by the second equation of (v) we may assume that the exponent is $+1$, and by the first equation of (v) we may assume that only the $H_{12}(a,b)$ occur. Let e_1 and e_2 denote the images in T of the identity elements of R and S, respectively. Then any element of T may be written uniquely as $re_1 + se_2 + j$, $r \in R$, $s \in S$, $j \in J$. First note that if $j, j' \in J$, then an application of (iv) using $a = j'$, $b = e_1$, $c = j$ yields

$$H_\alpha(j',j) = 1 \quad \text{for any } \alpha. \tag{3}$$

Now by applying (iii) twice and (3) once we obtain

$$H_\alpha(re_1 + se_2 + j', j) = H_\alpha(re_1, j)H_\alpha(se_2, j) \quad \text{for any } \alpha. \tag{4}$$

A similar equation is valid in case the variables are switched.

By applying (iv) with $a = re_1$, $b = j$, $c = e_2$, we obtain

$$H_\alpha(re_1,j) = H_\gamma(e_2,rj)^{-1}$$

and with $a = e_1$, $b = rj$, $c = e_2$, we obtain

$$H_\alpha(e_1,rj) = H_\gamma(e_2,rj)^{-1}. \tag{5}$$

These two equations yield

$$H_\alpha(re_1,j) = H_\alpha(e_1,rj) \tag{6}$$

and similarly one obtains

$$H_\alpha(se_2,j) = H_\alpha(e_2,js). \tag{7}$$

We can thus rewrite equation (4) as

$$H_\alpha(re_1 + se_2 + j',j) = H_\alpha(e_1,rj)H_\alpha(e_2,js). \tag{8}$$

As before, a similar result holds when the variables are switched. Also by applying the second part of (iii) we obtain

$$H_\alpha(e_i,j + j') = H_\alpha(e_i,j)H_\alpha(e_i,j'). \tag{9}$$

Applying (i), (iii) and (3) shows that

$$[H_\alpha(e_1,j), H_\alpha(e_2,j')] = 1. \tag{10}$$

Taking $r = 1$ in (5) and applying (6) and (ii) yields

$$H_\alpha(e_1,j) = H_\gamma(e_2,j)^{-1} = H_\gamma(-e_2,-j)^{-1} = H_{-\gamma}(j,e_2).$$

Next, (2) shows that ${}^{H_{-\gamma}(j,e_2)}w_\beta(1) = w_\beta(1)$ and thus using [**Mi**, Corollary 9.4],

$$H_{-\gamma}(j,e_2) = {}^{w_\beta(1)}H_{-\gamma}(j,e_2) = H_\alpha(j,e_2)$$

which yields

$$H_\alpha(e_1,j) = H_\alpha(j,e_2). \tag{11}$$

Similarly one obtains

$$H_\alpha(e_2,j) = H_\alpha(j,e_1). \tag{12}$$

To complete the proof that $K_2(n,J)$ is trivial, observe that by (8) we may assume that any element of $K_2(n,J)$ is a product of elements of the form $H_{12}(e_1,j)$, $H_{12}(e_2,j)$, $H_{12}(j,e_1)$, and $H_{12}(j,e_2)$. By (11) and (12) we may omit the last two of these. By (10) we can collect the elements of the two types, and by (9) we can combine them. Hence we may assume that any element of $K_2(n,J)$ is of the form $H_{12}(e_1,j_1)H_{12}(e_2,j_2)$. Upon applying φ we must obtain 1 and thus $j_1 = j_2 = 0$. Hence $K_2(n,J)$ is trivial, completing the proof of Theorem 1.

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