CONDITIONS FOR THE COMMUTATIVITY OF SEMIGROUPS

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ABSTRACT. Let S be a semigroup. Then by a theorem of Tully [7]: S is a commutative semigroup iff $ab = b^n a^m$ for all $a, b \in S$ ($m, n \geq 1$, fixed). We prove the following: S is a commutative semigroup iff $ab = b^{n(a,b)} a^{m(a,b)}$ for all $a, b \in S$, where one of the exponents $n(a,b)$ and $m(a,b)$ is constant and the other is independent of a or b .

Tamura formulated in [6] the following question: Let S be a semigroup and $f(x, y)$ a fixed word in $x, y \in S$; when does $ab = f(a, b)$ for all $a, b \in S$ imply the commutativity of S ? He proved the following result:

Let $f(a, b)$ be a word involving both a and b , and let the length of $f(a, b)$ be greater than 2. $ab = f(a, b)$ implies $ab = ba$ in semigroups iff

(i) $f(a, b)$ starts in b and ends in a , and

(ii) $ab = f(a, b)$ implies $ab = ba$ in groups.

Thus he obtained in particular: $ab = b^n a^m$ for all $a, b \in S$ and fixed $m, n \in N$ (N denotes the set of all natural numbers) implies the commutativity of S (Tully [7]).

Putchá and Weissglass [5] generalized Tamura's problem, supposing that $f(x, y)$ is not necessarily a fixed word but can vary with x and y . We prove the following

THEOREM. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is commutative.

(ii) $ab = b^n a^{m(a)}$

(iii) $ab = b^n a^{m(b)}$

(iv) $ab = b^{n(a)} a^m$

(v) $ab = b^{n(b)} a^m$

for all $a, b \in S$ and exponents in N (m, n are fixed and the other exponents depend only on the indicated variable).

REMARK. Note that conditions (iii) and (iv) and also (ii) and (v) are "dual": Assume for instance that condition (iii) holds in S . Then defining in S the dual operation " $\circ$ " (see e.g. Ljapin [2, p. 21]): $a \circ b = ba$, (iii) turns to $b \circ a = a^{m(b)} \circ b^n$, thus (iv) holds in $(S, \circ)$. Therefore we need only prove that (ii) resp. (iii) implies (i)—the other cases are trivial.

First we show the proposition (ii) resp. (iii) $\Rightarrow$ (i) for the special case of groups, which generalizes Satz 2 of Lausch, Nöbauer and Schweiger [3]:

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LEMMA. Let G be a group. Then the conditions (i)–(v) of the Theorem are equivalent.

PROOF. (ii) $\Rightarrow$ (i). Fixing $a = 1$ we obtain $b = b^n$, fixing $b = 1$ we get $a = a^{m(a)}$, together $ab = ba$.

(iii) $\Rightarrow$ (i). Setting $a = 1$ in (iii) we get $b = b^n$, thus

$$(1) \quad ab = ba^{m(b)}, \text{ hence } b^{-1}ab = a^{m(b)} \text{ for all } a, b \in G.$$

Now let U be an arbitrary subgroup of G ; then (1) implies $b^{-1}Ub = U$ for all $b \in G$. This means that every subgroup U of G is normal in G .

Suppose now that G is not abelian; then G is hamiltonian and contains the quaternion group Q as subgroup (see e.g. Hall [1, Chapter 12.5]). Evidently Q satisfies condition (iii), since G does. Q is a nonabelian group of order 8 defined by $g^4 = 1, g^2 = h^2, h^{-1}gh = g^{-1}$ —hence the order of the center of Q is 2. By assumption $g^{-1}xg = x^{m(g)}$ for all $x \in Q$. Taking $x = g$ we get $g = g^{m(g)}$, thus $m(g) \equiv 1 \pmod{4}$ ($= o(g) = \exp Q$). Therefore $g^{-1}xg = x$ for all $x \in Q$, which means $g \in Z(Q)$, which is a contradiction to $o(Z(Q)) = 2$. Thus G is abelian.

PROOF OF THE THEOREM. To show (ii) resp. (iii) $\Rightarrow$ (i) we can assume $n = 1$ in both cases, since otherwise we apply the theorem of Putcha and Weissglass [5] (see also the remark on p. 67) and obtain that S is an inflation of a semilattice of periodic groups G_i (for the definitions see, e.g., Petrich [4]). Since $G_i \subseteq S$ and the conditions (ii) and (iii) are hereditary to subsemigroups, we get by the Lemma that G_i is abelian for every i . Now an inflation of a semilattice of abelian groups is abelian again (see also Tamura [6]).

(ii) $\Rightarrow$ (i). Let S satisfy $ab = ba^{m(a)}$ for all $a, b \in S$. We state for arbitrary k

$$(2) \quad a^k b = ba^{km(a)} \text{ for all } a, b \in S.$$

The case $k = 1$ is verified by condition (ii), and now proceeding by induction we obtain

$$a^{k+1}b = a(ba^{km(a)}) = (ab)a^{km(a)} \stackrel{(ii)}{=} ba^{m(a)}a^{km(a)} = ba^{(k+1)m(a)}.$$

Now take an element c with $n(c) > 1$ and d arbitrary in S . Putting first $d = a, c = b$ and then $c = a, d^{m(d)} = b$ in condition (ii) we get

$$dc = cd^{m(d)} = d^{m(d)}c^{m(c)}.$$

Applying (2) with $d = a, m(d) = k, c^{m(c)} = b$ we obtain finally

$$dc = d^{m(d)}c^{m(c)} = c^{m(c)}d^{(m(d))^2}.$$

On the other hand, we get by condition (ii), taking first $c = a, d = b$ and then $d = a, c^{m(c)} = b$

$$cd = dc^{m(c)} = c^{m(c)}d^{m(d)}.$$

To conclude the proof we have only to show that $d^{m(d)} = d^{(m(d))^2}$ for all $d \in S$. Now fixing $a = b = d$ in condition (ii) we obtain $d^2 = d^{m(d)+1}$;

if $m(d) = 2$ it follows $d^2 = d^3 = d^4 = \dots = d^{m(d)} = \dots = d^{(m(d))^2}$. Now let $m(d) > 2$, then $d^2 = d^{m(d)+1}$ implies

$$d^{m(d)} = d^2 d^{m(d)-2} = d^{m(d)+1} d^{m(d)-2} = d^{2m(d)-1}$$

and also that $d^2, \dots, d^{m(d)}$ form a group with the unity $d^{m(d)-1}$, as is well known. Therefore we obtain

$$d^{m(d)} = d^{2m(d)-1} = d^{2m(d)-1} (d^{m(d)-1})^k \text{ for arbitrary } k.$$

Taking $k = m(d) - 1$ we get the desired result: $d^{m(d)} = d^{(m(d))^2}$. Thus S is commutative.

(iii) $\Rightarrow$ (i). Let S satisfy $ab = ba^{m(b)}$ for all $a, b \in S$. We define

$$I = \{b \in S, m(b) > 1 \text{ with } ab = ba^{m(b)}, \text{ for all } a \in S\}.$$

We show that I is a subsemigroup of S (it is easy to verify that I is even a prime ideal of S). Let $s, t \in I$ arbitrary, then

$$(3) \quad as = sa^{m(s)} \quad \text{and} \quad at = ta^{m(t)}$$

for all $a \in S$. We first state for arbitrary k and $a \in S$

$$(4) \quad (as)^k = s^k a^l \quad \text{with } l = m^k(s) + m^{k-1}(s) + \dots + m(s).$$

This is proved by induction on k ; the case $k = 1$ is verified by (3). Now

$$(as)^{k+1} = (s^k a^l)(as) = s^k (a^{l+1} s) \stackrel{(3)}{=} s^k (sa^{(l+1)m(s)}) = s^{k+1} a^{\bar{l}}$$

with $\bar{l} = (l+1)m(s) = m^{k+1}(s) + m^k(s) + \dots + m(s)$ concludes the proof of (4).

Applying equations (3) and (4) we get, therefore,

$$(5) \quad (as)t = t(as)^{m(t)} = ts^{m(t)} a^{m(s,t)}$$

with certain $m(s, t) > 1$. Now putting $a = s$ in the second equation of (3) we obtain $st = ts^{m(t)}$; together with (5) follows $ast = sta^{m(s,t)}$ with $m(s, t) > 1$ for all $a \in S$, hence I is a subsemigroup of S . By definition of I , therefore, $ab = ba^{m(b)}$ with $m(b) > 1$ specially for $a, b \in I$. Now we can apply the above mentioned theorem of Putcha and Weissglass [5] and obtain again that I is commutative. If we take finally an element $c \in S - I$, then c is in the center of S evidently; thus again S is commutative.

REMARK . The Theorem cannot be sharpened in the following sense: S is a commutative semigroup iff $ab = b^{n(a,b)} a^{m(a,b)}$ for all $a, b \in S$ with one exponent (even) constant and the other dependent on a and b . A counterexample is given by the quaternion group Q : since all subgroups of Q are normal, we get $g^{-1}ag = a^{m(a,g)}$ for all cyclic subgroups $\langle a \rangle$ specially. Consequently $ag = ga^{m(a,g)}$ is satisfied for all $a, g \in Q$, but Q is not abelian.

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