WEAK CONVERGENCE OF SEMIGROUPS
IMPLIES STRONG CONVERGENCE OF
WEIGHTED AVERAGES

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ABSTRACT. For a fixed p, $1 \leqslant p < \infty$, let $\{T_t: t > 0\}$ be a strongly continuous semigroup of positive contractions on L_p of a σ-finite measure space. We show that weak convergence of $\{T_t: t > 0\}$ in L_p is equivalent with the strong convergence of the weighted averages $\int_0^\infty T_t f \mu_n(dt)$ $(n \to \infty)$ for every $f \in L_p$ and every sequence (μ_n) of signed measures on $(0, \infty)$, satisfying $\sup_n \|\mu_n\| < \infty$; $\lim_n \mu_n(0, \infty) = 1$; and for each $d > 0$, $\lim_n \sup_{c > 0} |\mu_n|(c, c + d] = 0$. The positivity assumption is not needed if $p = 1$ or 2. We show that such a result can be deduced–not only in L_p, but in general Banach spaces–from the corresponding discrete parameter version of the theorem.

In recent years, various authors have studied the relations between weak and strong operator convergence: Blum-Hanson [4], Hanson-Pledger [7], Lin [9], Akcoglu-Sucheston [1], Jones-Kuftinec [8], Fong-Sucheston [6], and very recently Akcoglu-Sucheston [2], [3], who proved the theorem for positive L_p-contractions, $1 < p < \infty$. This theorem [1], [6], [3] states that if T is a positive contraction on L_p of a σ-finite measure space $(X, \mathcal{C}, m)$, where p is fixed and $1 \leqslant p < \infty$, then weak-$\lim_{n \to \infty} T^n f$ (w-$\lim_n T^n f$) exists for each $f \in L_p$ if, and only if, $\lim_{n \to \infty} \sum_{m=1}^\infty a_{nm} T^m f$ exists for every $f \in L_p$ and every matrix (a_{nm}) with real entries satisfying

$$(1.1) \qquad \sup_n \sum_m |a_{nm}| < \infty; \quad \lim_n \sum_m a_{nm} = 1; \quad \lim_n \max_m |a_{nm}| = 0.$$

It has also been shown that the positivity assumption is not needed if $p = 1$ or 2 [1], [6]. The problem of whether or not positivity is needed for $p \neq 1$ or 2 is still open. Matrices satisfying (1.1) were introduced in ergodic context in [6] and have been called *uniformly regular*; we denote the class of all uniformly regular matrices by $\mathfrak{A}_R$. Intuitively, a matrix $(a_{nm}) \in \mathfrak{A}_R$ if and only if it is *properly averaging*, in the sense that the masses a_{nm} spread as $n \to \infty$.

A semigroup $\{T_t: t > 0\}$, $T_t T_s = T_{t+s}$, of linear operators on a Banach space B is called *strongly continuous* if for each $x \in B$ and each $s > 0$, $\lim_{t \to s} \|T_t x - T_s x\| = 0$. R. Sato [10] recently obtained the following continuous parameter version of the strong ergodic theorem: For a fixed function f in L_2 and a strongly continuous semigroup $\{T_t: t > 0\}$ of contractions on L_2, w-$\lim_{t \to \infty} T_t f = f_0$ implies $\lim_{n \to \infty} \int_0^\infty a_n(t) T_t f \, dt = f_0$ for every sequence (a_n) of nonnegative, Lebesgue integrable functions on $(0, \infty)$ satisfying

$\int_0^\infty a_n(t) = 1$ and $\lim_{n\to\infty} \|a_n\|_\infty = 0$. In this note, we show that a stronger result can be deduced–not only in L_2, but in general Banach spaces–from the corresponding discrete version of the theorem (§2). In §3, we obtain as corollaries the continuous parameter version of the Akcoglu-Sucheston theorem for the Banach spaces $L_p(X, \mathcal{Q}, m)$, $1 \leqslant p < \infty$.

2. A linear operator T on a Banach space B is called *power-bounded* if $\sup_n \|T^n\| < \infty$; a semigroup $\{T_t : t > 0\}$ of linear operators on B is called *uniformly bounded* if $\sup_{t>0} \|T_t\| < \infty$. The *total variation* of a signed measure μ is denoted by $|\mu|$. We denote by $\mathfrak{A}$ the family of all sequences (μ_n) of signed measures on the σ-algebra of Lebesgue measurable subsets of $(0, \infty)$ satisfying

$$\sup_n \|\mu_n\| < \infty; \quad \lim_n \mu_n(0, \infty) = 1;$$

(2.1)

$$\lim_{n\to\infty} \sup_{c\geqslant 0} |\mu_n|(c, c + d] = 0 \quad \text{for each } d > 0.$$

REMARK 1. If (a_n) is a sequence of Lebesgue integrable functions on $(0, \infty)$ satisfying

$$\sup_n \int_0^\infty |a_n(t)| \, dt < \infty; \quad \lim_n \int_0^\infty a_n(t) \, dt = 1;$$

(2.2)

$$\lim_{n\to\infty} \sup_{c\geqslant 0} \int_c^{c+d} |a_n(t)| \, dt = 0$$

for each $d > 0$, and if we set $d\mu_n = a_n \, dt$, then $(\mu_n) \in \mathfrak{A}$. We note that sequences (a_n) satisfying (2.2) include those considered by Sato in [10].

REMARK 2. Let $\delta(t)$ denote the unit point mass at t. If $t_m > 0$, $t_m \to \infty$, $(a_{nm}) \in \mathfrak{A}_R$, and if we set $\mu_n = \sum_m a_{nm} \delta(t_m)$, then $(\mu_n) \in \mathfrak{A}$.

REMARK 3. If $x(t)$ is a bounded continuous function from $(0, \infty)$ to a Banach and if we set $d\mu_n = a_n \, dt$, then $(\mu_n) \in \mathfrak{A}$. We note that sequences (a_n) satisfying (2.2) include those considered by Sato in [10].

$$\left\| \int_0^\infty x(t)\mu(dt) \right\| \leqslant \left(\sup_{t\geqslant 0} \|x(t)\| \right) \cdot \|\mu\| \quad \text{(cf. [5]).}$$

THEOREM 2.1. *Let x be a fixed element in a Banach space B, real or complex. Then (α) implies (β):*

(α) For every power bounded linear operator T on B, if $\text{w-}\lim_{n\to\infty} T^n x = x_0$, then $\lim_{n\to\infty} \sum_{m=1}^\infty a_{nm} T^m x = x_0$ for every matrix $(a_{nm}) \in \mathfrak{A}_R$.

(β) For every uniformly bounded semigroup $\{T_t : t > 0\}$ of linear operators on B for which $T_t x$ is continuous on $(0, \infty)$, if $\text{w-}\lim_{t\to\infty} T_t x = x_0$, then $\lim_n \int_0^\infty T_t x \mu_n(dt) = x_0$ for every sequence $(\mu_n) \in \mathfrak{A}$.

The conclusion remains valid if "power bounded" and "uniformly bounded" in (α) and (β) are both replaced by "contraction".

PROOF. Let x be a fixed element in B, and assume that (α) holds for x. Let $\{T_t : t > 0\}$ be a semigroup satisfying the hypotheses of (β) and $(\mu_n) \in \mathfrak{A}$. We shall show that $\lim_n \int_0^\infty T_t x \mu_n(dt) = x_0$.

Let $\epsilon > 0$. The continuity of $T_t x$ on $[1, 2]$ implies that $T_t x$ is uniformly continuous on $[1, 2]$. Thus there is a positive integer k such that if

$$g(t) = \sum_{j=1}^{k} 1_{(1+(j-1)/k,\,1+j/k)}(t) \cdot T_{1+j/k}\, x,$$

then $\|g(t) - T_t x\| < \varepsilon$ for $t \in (1, 2]$. Here 1_A denotes the function that is 1 on A, and 0 elsewhere. Set $M = \sup_{t>0}\|T_t\|$, $K = \sup_n \|\mu_n\|$, and $I_i = (i, i + 1]$. Since $|\mu_n|(I_0) \to 0$ by (2.1), we have

$$\limsup_{n\to\infty} \left\| \int_0^\infty T_t x \mu_n(dt) - \sum_{i=0}^\infty \int_{I_{i+1}} T_i g(t - i)\mu_n(dt) \right\|$$

$$= \limsup_{n\to\infty} \left\| \int_{I_0} T_t x \mu_n(dt) + \sum_{i=0}^\infty \int_{I_{i+1}} (T_t x - T_i g(t - i))\mu_n(dt) \right\|$$

(2.3)
$$\leqslant \limsup_{n\to\infty} \left[M \cdot \|x\| \cdot |\mu_n|(I_0) \right.$$

$$\left. + \sum_{i=0}^\infty \|T_i\| \cdot \sup_{t\in I_1} \|T_t x - g(t)\| \cdot |\mu_n|(I_{i+1}) \right]$$

$$\leqslant M \cdot K \cdot \varepsilon.$$

For each $i \geqslant 0$, $1 \leqslant j \leqslant k$, set $I_{i,j} = (i + (j - 1)/k, i + j/k]$. It follows from the definition of $g(t)$ that for $n \geqslant 1$,

(2.4)
$$\sum_{i=0}^\infty \int_{I_{i+1}} T_i g(t - i)\mu_n(dt) = \sum_{i=0}^\infty \sum_{j=1}^{k} \mu_n(I_{i+1,j}) \cdot T_{i+1+j/k}\, x$$

$$= \sum_{m=k+1}^\infty a_{n,m} T^m x,$$

where $T = T_{1/k}$, and for $m = (i + 1)k + j$, $i \geqslant 0$, $1 \leqslant j \leqslant k$, $a_{n,m} = \mu_n(I_{i+1,j})$. It is easily checked that $(a_{n,m}) \in \mathfrak{A}_R$ since $(\mu_n) \in \mathfrak{A}$. Moreover, since $\{T_t: t > 0\}$ is uniformly bounded and $\text{w-lim}_{t\to\infty} T_t x = x_0$, we have that T is power bounded and $\text{w-lim}_{m\to\infty} T^m x = x_0$. Thus it follows from ($\alpha$) that $\lim_{n\to\infty} \sum_{m>k} a_{nm} T^m x = x_0$. Together with (2.3) and (2.4), we obtain that

$$\limsup_{n\to\infty} \left\| \int_0^\infty T_t x \mu_n(dt) - x_0 \right\| \leqslant MK\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, (β) holds.

It is clear that the second part of the theorem can be proved in the same way. $\square$

COROLLARY 2.1. *Let B be a Banach space. Then $(\alpha)'$ implies $(\beta)'$:*

$(\alpha)'$ For every power bounded linear operator T on B, if $\text{w-lim}_{n\to\infty} T^n x$ exists for every $x \in B$, then $\lim_{n\to\infty} \sum_{m=1}^\infty a_{n,m} T^m x$ exists and is equal to $\text{w-lim}_{n\to\infty} T^n x$ for every $(a_{n,m}) \in \mathfrak{A}_R$.

$(\beta)'$ For every strongly continuous, uniformly bounded semigroup $\{T_t: t > 0\}$ of linear operators on B, if $\text{w-lim}_{t\to\infty} T_t x$ exists for every $x \in B$, then $\lim_n \int_0^\infty T_t x \mu_n(dt)$ exists for every sequence $(\mu_n) \in \mathfrak{A}$, and is equal to $\text{w-lim}_{t\to\infty} T_t x$.

The conclusion remains valid if "power bounded" and "uniformly bounded" are both replaced by "contraction".

PROOF. Immediate from Theorem 2.1. □

We next show that the converse of statement (β) in Theorem 2.1 is valid in general Banach spaces.

PROPOSITION 2.1. *Let x be a fixed element in a Banach space B, real or complex. Let $\{T_t: t > 0\}$ be continuous linear operators on B such that the vector-valued function $T_{(\cdot)} x$ from $(0, \infty)$ to B is continuous and $\sup_{t>0} \|T_t x\| < \infty$. Then (b) implies (a):*

(a) *w-$\lim_{t \to \infty} T_t x$ exists.*
(b) *$\lim_{n \to \infty} \int_0^\infty T_t x \mu_{n(dt)}$ exists for every sequence $(\mu_n) \in \mathfrak{A}$.*

PROOF. We first consider the case where B is a real Banach space. Assume that (b) holds but (a) fails. Then there exists an $x^* \in B^*$ such that $h(t) = \langle T_t x, x^* \rangle$ diverges as $t \to \infty$, where B^* is the dual space of B. Since

$$\sup_{t>0} |h(t)| \leqslant \sup_{t>0} \|x^*\| \, \|T_t x\| < \infty,$$

h is bounded on $(0, \infty)$. $h(t)$ is also continuous on $(0, \infty)$ since $T_t x$ is. Thus, $h(t)$ being divergent as $t \to \infty$, there are constants α, β with $\alpha < \beta$, and a sequence (t_i) with $t_i \uparrow \infty$, such that $h(t_i) \geqslant \beta$ if i is odd, and $h(t_i) \leqslant \alpha$ if i is even. Set for $n \geqslant 1$,

$$\mu_{2n} = \frac{1}{n} \sum_{k=1}^n \delta(t_{2k}), \quad \mu_{2n-1} = \frac{1}{n} \sum_{k=1}^n \delta(t_{2k-1}),$$

where $\delta(t)$ denotes the unit point mass at t. Then $(\mu_n) \in \mathfrak{A}$, but

$$\liminf_n \left\langle \int_0^\infty T_t x \mu_n(dt), x^* \right\rangle = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n h(t_{2k})$$

$$\leqslant \alpha < \beta \leqslant \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^n h(t_{2k-1})$$

$$= \limsup_{n \to \infty} \left\langle \int_0^\infty T_t x \mu_n(dt), x^* \right\rangle.$$

Hence $\left(\int_0^\infty T_t x \mu_n(dt) \right)_{n=1}^\infty$ does not converge weakly, and *a fortiori,* strongly.

If B is a complex Banach space, then either the real part or the imaginary part of $h(t)$ diverges as $t \to \infty$, and can be used to replace $h(t)$ in the above argument. □

REMARK 4. The vector-valued function $T_t x$ in Proposition 3.1 may be replaced by any vector-valued function $x(t)$ from $(0, \infty)$ to B such that $x(t)$ is continuous and bounded on $(0, \infty)$.

3. We now apply the results in §2 to the Banach spaces L_p of a σ-finite measure space $(X, \mathcal{C}, m)$, $1 \leqslant p < \infty$. An operator T on L_p is called *positive* if $Tf \geqslant 0$ whenever $f \geqslant 0$. Theorem 3.1 below strengthens the result of R. Sato mentioned in §1.

THEOREM 3.1. *Let $\{T_t: t > 0\}$ be a contraction semigroup on $L_2(X, \mathcal{C}, m)$, and let f be a fixed function in L_2 such that $T_t f$ is continuous on $(0, \infty)$. Then conditions (a) and (b) are equivalent:*

(a) w-lim$_{t\to\infty} T_t f = f_0$.

(b) lim$_n \int_0^\infty T_t f \mu_n(dt) = f_0$ for every $(\mu_n) \in \mathfrak{A}$.

PROOF. This follows from Theorem 1.1 in [6], Proposition 2.1 and Theorem 2.1. □

THEOREM 3.2. *Let* $\{T_t: t > 0\}$ *be a strongly continuous contraction semigroup on* $L_1(S, \mathfrak{C}, m)$. *Then conditions* (A) *and* (B) *are equivalent*:

(A) *For each* $f \in L_1$, w-lim$_{t\to\infty} T_t f$ *exists*.

(B) *For each* $f \in L_1$, lim$_{n\to\infty} \int_0^\infty T_t f \mu_n(dt)$ *exists for every* $(\mu_n) \in \mathfrak{A}$, *and is equal to* w-lim$_{t\to\infty} T_t f$.

PROOF. This follows from Theorem 1.3 in [6], Proposition 2.1, and Corollary 2.1. □

THEOREM 3.3 *Let* $\{T_t: t > 0\}$ *be a strongly continuous semigroup of positive contractions on* $L_p(X, \mathfrak{C}, m)$, *where* p *is fixed*, $1 < p < \infty$. *Then conditions* (A) *and* (B) *are equivalent*:

(A) *For each* $f \in L_p$, w-lim$_{t\to\infty} T_t f$ *exists*.

(B) *For each* $f \in L_p$, lim$_{n\to\infty} \int_0^\infty T_t f \mu_n(dt)$ *exists for each sequence* $(\mu_n) \in \mathfrak{A}$, *and is equal to* w-lim$_{n\to\infty} T_t f$.

PROOF. We observe that the conclusions in Theorem 2.1 and Corollary 2.1 remain valid if $B = L_p(X, \mathfrak{C}, m)$, and "power bounded" and "uniformly bounded" in (α) and (β) are both replaced by "positive contraction". Theorem 3.3 now follows from Theorem 1.4 in [3] and Proposition 2.1. □

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