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HYPERSPACES OF HEREDITARILY INDECOMPOSABLE PLANE CONTINUA

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ABSTRACT. In this note we prove that if X is a hereditarily indecomposable plane continuum then the hyperspace $C(X)$ can be embedded in Euclidean 3-space.

If X is a continuum, we let $C(X)$ denote the hyperspace of subcontinua of X with the Hausdorff metric. A continuous function μ from $C(X)$ into the reals R is called a Whitney map if the following two conditions are satisfied:

(i) $A \subset B$ and $A \neq B \Rightarrow \mu(A) < \mu(B)$.

(ii) $\mu(\{x\}) = 0$ for each $x \in X$.

Whitney maps always exist [4].

The aim of this note is to prove the following:

THEOREM. *If X is a hereditarily indecomposable plane continuum, then $C(X)$ is embeddable in Euclidean 3-space R^3.*

Transue [3] proved this theorem for the case in which X is a continuum which does not separate the plane. Krasinkiewicz [2] proved that $C(X)$ is embeddable in R^4. J. T. Rogers recently announced the theorem in case X separates the plane into finitely many disjoint regions. I am grateful to Professor Krasinkiewicz for pointing out this problem to me. The argument that is given here is a modification of that given by Transue [3].

PROOF. We suppose without loss of generality that X is contained in the 2-sphere S^2. Let $p \in X$. Let P denote the set of continua in X which contain p. We prove that P is an arc in $C(X)$.

It is clear that P is compact. If P is not connected then P can be written as the union of two mutually separated sets H and K. We may suppose $X \in H$. Since K is compact there is a maximal element T of K (i.e. T is contained in no other element of K). Then T is in the closure of H for if U is any neighborhood of T in X, then the closure of the component of U which contains T is an element of P which properly contains T and hence is in H. This is a contradiction since H and K are separated sets. We have proved that P is a continuum.

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If S, $T \in P$ then either $S \subset T$ or $T \subset S$ since X is hereditarily indecomposable and $p \in S \cap T$. Thus, P is totally ordered under inclusion.

Let μ: $C(X) \to R$ be a Whitney map. It follows from the above argument that $\mu|_P$ carries P homeomorphically onto the interval $[0, \varepsilon]$ where $\varepsilon = \mu(X)$.

Let $\sim$ be the equivalence relation defined on $X \times [0, \varepsilon]$ whose equivalence classes are the sets $g \times \{\mu(g)\}$ for $g \in C(X)$. To see that $\sim$ is upper semicontinuous, let $(x_i, a_i))$ and $(y_i, a_i))$ be sequences in $X \times [0, \varepsilon]$ such that for each i, $(x_i, a_i) \sim (y_i, a_i)$ and $\lim x_i = x$, $\lim y_i = y$ and $\lim a_i = a$. For each i there exists $g_i \in C(X)$ such that $\mu(g_i) = a_i$ and $x_i, y_i \in g_i$. Without loss of generality we may suppose the sequence $g_i)$ converges to g in the compact space $C(X)$. Since μ is continuous $\mu(g) = \lim \mu(g_i) = \lim a_i = a$. Clearly, $x, y \in g$ so $(x, a) \sim (y, a)$ and $\sim$ is upper semicontinuous.

Let U_1, $U_2, \ldots$ be the complementary components of X in S^2. For each $k \in C(X)$ let $k^* = k \cup \bigcup\{U_i | \mathrm{Bd}\,(U_i) \subset k\}$ where $\mathrm{Bd}\,(U_i)$ denotes the boundary of U_i.

We show that

$$(\dagger) \qquad k^* = k \cup \bigcup\{W | W \text{ is a component of } S^2 \backslash k \text{ and } W \subset S^2 \backslash X\}.$$

If W is a component of $S^2\backslash k$ then $\mathrm{Bd}\,(W) \subset k$ since k is closed and W is open (since S^2 is locally connected). If $W \subset S^2\backslash X$ then W is a component of $S^2\backslash X$ and so $W \subset k^*$. Let $U_i \subset k^*$. Then $\mathrm{Bd}\,(U_i) \subset k$. Now, U_i is a component of $S^2\backslash X$. Since $\mathrm{Bd}\,(U_i) \subset k$, U_i is also a component of $S^2\backslash k$. This completes the proof of (†).

Since X is indecomposable, $X\backslash k$ is connected. It follows that at most one component of $S^2\backslash k$ meets X and so $S^2\backslash k^*$ is connected and open. Thus, k^* is closed. It is clear from the definition of k^* that k^* is connected. We have proved that k^* is a continuum which does not separate S^2.

We extend $\sim$ to an equivalence relation $\sim^*$ on $S^2 \times [0, \varepsilon]$ where the equivalence classes of $\sim^*$ are points and the sets $g^* \times \{\mu(g)\}$ where $g \in C(X)$. For each $a \in [0, \varepsilon]$ the members of $\mu^{-1}(a)$ are pairwise disjoint so the sets $g^* \times \{\mu(g)\}$ are also pairwise disjoint by the definition of g^*. It remains to prove that $\sim^*$ is upper semicontinuous.

Let $(x_i, a_i))$ and $(y_i, a_i))$ be sequences in $S^2 \times [0, \varepsilon]$ such that $(x_i, a_i))$ converges to (x, a), $(y_i, a_i))$ converges to (y, a) and for each i, $(x_i, a_i) \sim^* (y_i, a_i)$. If $x_i = y_i$ for infinitely many i, then $x = y$ and $(x, a) \sim^* (y, a)$. We suppose, therefore, that for each $i, x_i \neq y_i$. For each i let $g_i \in C(X)$ such that $x_i, y_i \in g_i^*$ and $\mu(g_i) = a_i$. We may also suppose that the sequence $g_i)$ converges to g in $C(X)$. Since μ is continuous, $\mu(g) = \lim \mu(g_i) = \lim a_i = a$. If for each i, $x_i \in X$, then $x_i \in g_i$ and $x \in g$. Let us suppose now that for each i, $x_i \notin X$. For each i, let U_{j_i} be the component of $S^2\backslash X$ which contains x_i. Then $\mathrm{Bd}\,(U_{j_i}) \subset g_i$. If $x \in X$ then either $x \in g_i$ or g_i separates x and x_i in S^2. In either case $x \in \lim g_i = g$. If $x \in U_k$ for some k then for all sufficiently large i, $x_i \in U_k$ and $U_{j_i} = U_k$. Hence $\mathrm{Bd}\,(U_{j_i}) = \mathrm{Bd}\,(U_k) \subset g_i$ for all sufficiently large i. In particular $\mathrm{Bd}\,(U_k) = \lim \mathrm{Bd}\,(U_{j_i}) \subset \lim g_i = g$. In all cases $x \in g^*$. Similarly, $y \in g^*$ and so $(x, a) \sim^* (y, a)$. This completes the proof that $\sim^*$ is upper semicontinuous.

Let π be the natural projection of $S^2 \times [0, \varepsilon]$ onto the quotient space

$(S^2 \times [0,\varepsilon])/\sim^*$. Let h: $(S^2 \times [0,\varepsilon])/\sim^* \to [0,\varepsilon]$ be such that $h(\pi(x,a)) = a$. Then h is 0-regular as in the proof of Theorem 8 in [1]. By Moore's theorem $h^{-1}(\delta)$ is a 2-sphere for each δ such that $0 \leqslant \delta < \varepsilon$ and $h^{-1}(\varepsilon)$ is a point. It follows from Theorem 7 in [1] that $(S^2 \times [0,\varepsilon])/\sim^*$ is a 3-cell. This completes the proof of the theorem for $(X \times [0,\varepsilon])/\sim$ is homeomorphic to $C(X)$ and $(X \times [0,\varepsilon])/\sim$ is embedded in the 3-cell $(S^2 \times [0,\varepsilon])/\sim^*$.

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