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SYMMETRIC OVERMAPS

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ABSTRACT. We prove periodicity theorems for the degrees of fibre-preserving maps of sphere bundles, and of projective space bundles.

This note is our first on the subject of fibre-preserving maps, called *overmaps*, and comes from [6]. I wish to thank my supervisor, Professor I. M. James, for encouragement and for considerable help with the exposition.

Let M, M' be connected compact oriented q-manifolds, where $q \geqslant 1$. For $K \geqslant 1$, let a transitive permutation group Γ on K letters permute the factors of M^K. A map from M^K is *symmetric* when it is constant on the orbits of Γ. *The degree* of a symmetric map is the Brouwer degree of its restriction $M \to M'$ to a factor.

Constant maps are symmetric and, if $K = 1$, all maps from M to M' are symmetric. If M is a rational cohomology sphere then, by [2], a necessary condition for there to be a symmetric map $M^K \to M'$ of nonzero degree is that q be odd or $K = 1$.

Let E, E' be oriented fibre bundles over a path-connected space B with fibres M, M'. We denote the fibre product $E \times_B E \times_B \cdots E$ (K factors) by $E^{(K)}$. An overmap $E^{(K)} \to E'$ is *symmetric of degree m* when its restriction to fibres is a symmetric map of degree m. In particular, if $K = 1$, all overmaps from E to E' are symmetric.

Let a group G act fibrewise on E, with the product action on $E^{(K)}$, and fibrewise orthogonally on an oriented orthogonal q-sphere bundle F over B. When q is odd we orient the real projective q-space bundle PF associated with F, and we let G act, so that the identification overmap h: $F \to PF$ is G-invariant of degree 2.

THEOREM 1. *Let q be odd, let E' be F or PF, and let n be the degree of a G-invariant symmetric overmap from $E^{(K)}$ to E'. There is an integer $\alpha_K(E, E') \geqslant 0$ such that there is a G-invariant symmetric overmap $E^{(K)} \to E'$ of degree m if and only if $m \equiv n$* mod α.

Taking $E = E'$, $K = n = 7$ in Theorem 1, we obtain the following result.

COROLLARY 2. *Let q be odd, and let E' be F or PF. There is an integer $\alpha(E') \geqslant 0$ such that there is a G-invariant overmap of degree m from E' to itself if and only if $m \equiv 1$* mod α.

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In [7] we take G to be trivial, and we show that as E' varies $\alpha(E')$ takes all nonnegative integer values. In general, $\alpha(PF) \neq \alpha(F)$. For example, let G be trivial, and let F be the Whitney multiple $(q+1)h$ of the identification $h\colon S^n \to RP^n$, where S^n, RP^n are the n-sphere, projective real n-space. Then $\alpha(F) = 1, 2$ according as $n \leqslant q, n > q$. However, PF is trivial and, by Corollary 2, $\alpha(PF) = 1$.

Let the fibre bundles E, E' be *ex-spaces* [4]. Thus E, E' are equipped with cross-sections s, s' which we suppose G-invariant. Also an overmap $f\colon E \to E'$ is an *ex-map* when $fs = s'$. We regard $E^{(K)}$ as an ex-space by means of the cross-section $s \times_B s \times_B \cdots s$. Let the sphere bundle F be an ex-space, and regard PF as an ex-space by requiring $h\colon F \to PF$ to be an ex-map.

THEOREM 3. *Let q be odd, and let E' be F or PF. There is an integer $\beta_K(E, E')$ such that there is a G-invariant symmetric ex-map $E^{(K)} \to E'$ of degree m if and only if $m \equiv 0 \bmod \beta$.*

Taking $E = E'$, $K = 1$ in Theorem 3, we obtain the following result.

COROLLARY 4. *Let q be odd, and let E' be F or PF. There are G-invariant ex-maps of all degrees from E' to itself.*

When B is a connected compact oriented manifold, so are E, E', and the degree of an overmap from E to E' is its Brouwer degree. Hence, and by fibre-suspension, Corollary 4 generalizes [8, 1.5].

Let G be trivial, and let B be a connected finite CW-complex. When q is odd, and E' is F or PF, then $\beta_K(E, E')$ depends on the vertical homotopy classes of s, s', whereas $\alpha_K(E, E')$ evidently does not. For example, let $E = E' = B \times S^q$, $B = S^r$, and let s, s' correspond to $0, \nu \in \pi_r S^q$. Then $\beta_1(E, E')$ is the order of the Whitehead product $[\nu, \iota_q]$, where ι_q generates $\pi_q S^q$. However, $\alpha_1(E, E') = 1$.

Because of the main result of [1], the argument of [3, §2] also applies to real projective spaces. Corollary 4 then allows us to argue fibrewise, proving the following generalization of [3, 2.3].

COROLLARY 5. *Let q be odd, and let E' be F or PF. Then $\beta_K(E, E')$ is positive. Further, no prime factor of $\beta_K(F, E')$ or of $\beta_K(PF, E')$ exceeds K.*

Let B be a point. Taken with Corollary 5, Theorem 3 generalizes [3, 1.2] to include symmetric maps of projective spaces. By [5] $\beta_2(S^q, S^q)$ is $2^{(q+1)/2}$ or $2^{(q-1)/2}$ according as $q \equiv 3, 5$ or $q \equiv 1, 7 \bmod 8$.

To prove Theorems 1, 3, let $O(q+1)$ denote the group of orthogonal transformations of S^q. Let s, t be integers, and define an $O(q+1)$-map $k_t'\colon S^q \times S^q \to S^q$ as follows.

$$\begin{aligned} &k_t'(x,y) = (x \sin(1-t)\theta + y \sin t\theta) \operatorname{cosec} \theta, \\ (1) \qquad &k_t'(x,x) = x, \quad k_t'(x,-x) = (-1)^t x, \end{aligned}$$

where $x, y \in S^q$, $x \neq \pm y$, and $0 < \theta < \Pi$ is chosen so that $\cos\theta$ is the Euclidean inner product $(x \cdot y)$.

For $x, y \in S^q$ we have the following identities.

(2) $$k'_t(x,y) = k'_{1-t}(y,x),$$

(3) $$k'_{st}(x,y) = k'_s(x, k'_t(x,y)),$$

(4) $$k'_t(x,-y) = (-1)^t k'_t(x,y),$$

(5) $$k'_{-1}(x,y) = T_{q+2}(x)(j'_{-1}(y)),$$

where $T_{q+2}\colon S^q \to O(q+1)$ is the characteristic map [9, §23.4] for the tangent bundle to S^{q+1}, and where $j'_{-1}\colon S^q \to S^q$ is the suspension of the antipodal map on the hyperplane orthogonal to $p^q = (0,0,\ldots,1) \in S^q$.

We may also describe k'_t as follows. Given $x, y \in S^q$, let θ be the distance along a geodesic from x to y. On this geodesic, and at distance $t\theta$ from x, we have $k'_t(x,y)$. From this description, or by induction on t using (2), (3), (5), k'_t is continuous. We denote $k'_t|\{p^q\} \times S^q$ by $j_t\colon S^q \to S^q$.

(6) According as q is odd or even, j_t has degree t or $(1 + (-1)^{t-1})/2$.

To prove (6), note that if $t > 0$ then $j_t = 1 + a + 1 + \cdots$ (t summands), where 1, a denote the identity, the antipodal map on S^q, and where '+' means head to tail addition along the p^q axis. Since a has degree $(-1)^{q+1}$ this proves (6) for $t > 0$. But $j_{-t} = j_{-1}j_t$ by (3), and j_{-1}, j_0 have degrees $(-1)^q$, 0. This completes the proof of (6).

By (4), (2), k'_t respects the identification $h\colon S^q \to RP^q$, and therefore projects to an $O(q+1)$ - map $RP^q \times RP^q \to RP^q$ which we also refer to as k'_t. Let q be odd. By (6), (2), and since h is of degree 2, we have the following assertion.

(7) The restriction of k'_t to the first, second factor has degree $1 - t$, t.

In the situation of Theorem 1, k'_t extends from fibres to a G-invariant overmap $g\colon E' \times_B E' \to E'$. If E' is an ex-space then g is an ex-map, since $k'_t(x,x) = x$ by (1).

(8) Let $\Delta\colon E^{(K)} \to E^{(K)} \times_B E^{(K)}$ be the diagonal overmap. Given G-invariant symmetric overmaps $f_i\colon E^{(K)} \to E'$ of degrees m_i ($i = 1,2$), the composite $g(f_2 \times_B f_1)\Delta$ is a G-invariant symmetric overmap of degree $tm_1 + (1-t)m_2$.

Taken with the following remark, (8) proves Theorem 1.

(9) Let A be a nonempty set of integers such that, if $m_1, m_2 \in A$ then, for all integers t, $tm_1 + (1-t)m_2 \in A$. Then, for some integers n, α, $A = \{m\colon m \equiv n \bmod \alpha\}$.

In the situation of Theorem 3, we may read 'ex-map' for 'overmap' in (8). Taken with (9), this proves Theorem 3.

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