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A REMARK ON A RESULT OF MCKEAN

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ABSTRACT. The diameter of the Dirichlet polygon associated to certain discontinuous groups acting on the upper half-plane is shown to be bounded. This clarifies a point in the proof of a result of McKean.

In a very useful paper [2] McKean proves the following interesting result: *There are, up to isometry, only finitely many compact Riemann surfaces M corresponding to a given spectrum of the Laplacian on M.* Here we are regarding M as the quotient of the upper half-plane H^+ by a discontinuous group Γ of hyperbolic transformations and assuming that H^+ is endowed with the metric $((dx)^2 + (dy)^2)/y^2$.

There is a point in McKean's proof of this result which is not completely obvious, and it is the purpose of this note to give a simple way around this. The problem involves bounding the diameter of a fundamental polygon, S_Γ, for Γ in terms of the diameter of M itself. This can easily be circumvented by the following result.

THEOREM. *Let S_Γ be the Dirichlet polygon for Γ centered at i, i.e. the fundamental domain bounded by segments of perpendicular bisectors of the geodesics joining i and its translates by Γ. Then, provided the genus of M is fixed and there is a lower bound on the shortest closed geodesic (which is automatically furnished by the Selberg trace formula, when the spectrum is given), $\exists$ a constant $C > 0$ such that* $\operatorname{diam}(S_\Gamma) \leqslant C$ *for all* Γ.

We will prove the Theorem by contradiction. Suppose $\{\Gamma_r\}$ is a sequence of discontinuous groups consisting of hyperbolic transformations, each having compact S_{Γ_r} and such that $\operatorname{diam}(S_{\Gamma_r}) \to \infty$, and for all r the corresponding Riemann surfaces have the same fixed genus g, i.e. $\operatorname{meas}(S_{\Gamma_r}) = A$, where $A = 4\pi(g-1)$. Assume also that for all r, $\min l_{\gamma_r}$, the length of the shortest closed geodesic, is greater than ε for some $\varepsilon > 0$.

Then if $\gamma_r \in \Gamma_r$ is not the identity, $|\operatorname{sp}(\gamma_r)| \geqslant 2 + \eta(\varepsilon)$, where $|\operatorname{sp}(\gamma_r)|$ is the absolute value of the trace of γ_r (this is well defined for $\gamma \in \operatorname{PSL}(2, R)$), and $\eta(\varepsilon) = 2\cosh \varepsilon/2 - 2$. Let V be the set of transformations such that if $\gamma \in V$, then $|\operatorname{sp}(\gamma)| < 2 + \eta(\varepsilon)$. Clearly $\Gamma_r \cap V = e$ for all r (e is the identity element of $G = \operatorname{PSL}(2, R)$). Now our sequence of discontinuous groups satisfies the hypotheses of Theorem 1 and Lemma 7 of [1], which, adjusted to our case, state that: If $\{\Gamma_r\}$ is a sequence of lattices in G and if (1) $\exists$ an open neighborhood V of e such that $\Gamma_r \cap V = e$ for all r, and (2) $\exists$ a constant $A < \infty$ such that $\operatorname{meas}(G/\Gamma_r) \to A$, then one can extract from $\{\Gamma_r\}$ a

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subsequence $\{\Gamma_{r'}\}$ which converges to a lattice Γ with $\Gamma \cap V = e$ and meas $(G/\Gamma) = A$. Here $\Gamma_{r'} \to \Gamma$ means that if U is any neighborhood of e in G, and K is a compact set in G, then for r sufficiently large, to each $\gamma \in \Gamma_r \cap K$ there corresponds an $\alpha \in \Gamma$ such that $\alpha^{-1}\gamma \in U$, and for each $\alpha \in \Gamma \cap K$ there is $\gamma \in \Gamma_r$ such that $\alpha^{-1}\gamma \in U$. Thus our sequence $\{\Gamma_r\}$ has a subsequence $\{\Gamma_{r'}\}$ converging in this sense to a discontinuous group Γ, with $\Gamma \cap V = e$, and meas $(S_\Gamma) = A$.

Claim. S_Γ is compact. If not, Γ must admit parabolic transformations and this cannot happen since, apart from the identity, all $\gamma \in \Gamma$ satisfy $|\mathrm{sp}\,(\gamma)| \geqslant 2 + \eta(\varepsilon)$. Thus the diameter of S_Γ is bounded and if $\Gamma_r \to \Gamma$, diam $(S_{\Gamma_r}) \to$ diam (S_Γ). The last statement follows immediately from the definition of the limit of a sequence of discontinuous groups. Let $g_1, g_2, \ldots, g_k$ be the set of generators of Γ which give the arcs of S_Γ. Then for r large enough we can find $g_1^r, g_2^r, \ldots, g_k^r$ generators of Γ_r which give the arcs of S_{Γ_r} and $g_i^r \to g_i$, so we are done.

The bound on the diameter of a fundamental domain is used in McKean's paper to show that if $g_1, g_2, \ldots, g_n$ are generators of Γ, then $|\mathrm{sp}\,(g_i)|$, $|\mathrm{sp}\,(g_i g_j)|$, and $|\mathrm{sp}\,(g_i g_j g_k)|$ are bounded. This together with the fact that sp (g_i), sp $(g_i g_j)$ and sp $(g_i g_j g_k)$ determine Γ up to conjugation in PSL $(2, R)$ and/or reflection completes McKean's proof.

References

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