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A CHARACTERISTICALLY NILPOTENT LIE ALGEBRA CAN BE A DERIVED ALGEBRA

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ABSTRACT. An example is constructed of a Lie algebra whose derived algebra has only nilpotent derivations, thus answering a question of Dixmier and Lister.

1. **Introduction.** In a well-known paper in this journal [1], Dixmier and Lister constructed the first example of a characteristically nilpotent Lie algebra, that is, a Lie algebra with only nilpotent derivations. Upon proving that their example has the additional property that it is not the derived algebra of any Lie algebra, they pose the question: If L is any characteristically nilpotent Lie algebra, is it necessarily true that L cannot be a derived algebra? Their proof shows that the answer is yes if every derivation of L maps L into its derived algebra. Leger and Tôgô [2] have shown the answer to be yes under certain other conditions, for example, if every derivation of L annihilates the center of L. The purpose of this paper is to resolve Dixmier and Lister's question in the negative. We construct an 18-dimensional Lie algebra, H, whose derived algebra, $[H,H]$, is characteristically nilpotent.

2. **The example.** Let L denote the 16-dimensional Lie algebra over any field Φ, of characteristic not 2 or 5 with basis $\{x_1, x_2, \ldots, x_{16}\}$ and multiplication determined by

$$\begin{gathered}
[x_1,x_2] = x_7, \quad [x_1,x_3] = x_8, \quad [x_1,x_4] = x_9, \quad [x_1,x_5] = x_{10}, \\
[x_1,x_6] = x_{13}, \quad [x_1,x_7] = x_{15}, \quad [x_1,x_8] = x_{16}, \quad [x_2,x_3] = x_{11}, \\
[x_2,x_4] = x_{12}, \quad [x_2,x_5] = x_{15}, \quad [x_2,x_6] = x_{14}, \quad [x_2,x_7] = -x_{16}, \\
[x_3,x_4] = -x_{13} - (9/5)x_{15}, \quad [x_3,x_5] = -x_{14}, \quad [x_3,x_6] = -x_{16}, \\
[x_4,x_5] = 2x_{16}, \quad \text{and} \quad [x_i,x_j] = 0 \text{ for } i+j \geqslant 10.
\end{gathered}$$

Note that for distinct i, j, k, the products $[[x_i,x_j],x_k]$ are all 0; thus the Jacobi identity is immediately verified.

REMARK. Recall that an ideal in any Lie algebra, G, is called *characteristic* if it is invariant under all derivations of G. Note that, if I and J are characteristic ideals in G, then so are $[I,J]$ and the transporter of I to J, i.e., $\{x \in G \| [x,I] \subset J\}$.

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PROPOSITION 1. *L is characteristically nilpotent.*

PROOF. For $1 \leqslant i \leqslant 16$, let I_i denote the ideal of L spanned by $\{x_j\}_{j \geqslant i}$. The following five statements show that I_2, I_3, I_4, I_5, I_6 are characteristic:

(i) I_3 is the transporter of $[L, L](= I_7)$ to 0.

(ii) I_4 is the transporter of L to the center of $L(= I_9)$.

(iii) I_2 is the transporter of $[L, L]$ to $[I_4, I_4](= I_{16})$.

(iv) I_6 is the transporter of L to $[I_2, I_2](= I_{11})$.

(v) I_5 is the transporter of I_2 to $[L, I_6](= I_{13})$.

Now let D be a derivation of L. We shall show D is nilpotent. Since $I_2, \ldots, I_6$ are characteristic (and $I_1 = L$),

$$D(x_i) \equiv c_i x_i \bmod (I_{i+1}), \quad \text{for } 1 \leqslant i \leqslant 6, \text{ where } c_i \in \Phi.$$

Using $[x_1, x_2] = x_7$ and $[x_1, I_3] + [I_2, x_2] \subset I_8$, we get

$$D(x_7) = [Dx_1, x_2] + [x_1, Dx_2] \equiv (c_1 + c_2)x_7 \bmod (I_8).$$

Similarly $[x_1, x_3] = x_8$ and $[x_1, I_4] + [I_2, x_3] \subset I_9$ imply

$$D(x_8) \equiv (c_1 + c_3)x_8 \bmod (I_9).$$

Then

$$D(x_{16}) = D(-[x_3, x_6]) = (c_3 + c_6)x_{16},$$
$$D(x_{16}) = D(-[x_2, x_7]) = (c_1 + 2c_2)x_{16},$$
$$D(x_{16}) = D([x_1, x_8]) = (2c_1 + c_3)x_{16},$$

and

$$D(x_{16}) = D(\tfrac{1}{2}[x_4, x_5]) = (c_4 + c_5)x_{16},$$

so $c_3 + c_6 = c_1 + 2c_2 = 2c_1 + c_3 = c_4 + c_5$. Next

$$D(x_{15}) = D([x_1, x_7]) \equiv (2c_1 + c_2)x_{15} \bmod (I_{16}).$$

In particular, this last relation implies that $D(x_{15})$ has no x_{14} component, which we use, noting $[I_3, x_5] + [x_2, I_6] \subset (x_{14}, x_{16})$, to get

$$D(x_{15}) = D([x_2, x_5]) \equiv (c_2 + c_5)x_{15} \bmod (I_{16}).$$

Thus $2c_1 + c_2 = c_2 + c_5$. Also,

$$D(x_{14}) = D([x_2, x_6]) \equiv (c_2 + c_6)x_{14} \bmod (I_{15}),$$

and

$$D(x_{14}) = D(-[x_3, x_5]) \equiv (c_3 + c_5)x_{14} \bmod (I_{15}),$$

so $c_2 + c_6 = c_3 + c_5$. Finally,

$$D(x_{13}) = D([x_1, x_6]) \equiv (c_1 + c_6)x_{13} \bmod (I_{14})$$

and

$$D(x_{13}) = D(-[x_3, x_4] - (9/5)x_{15}) \equiv (c_3 + c_4)x_{13} \operatorname{mod}(I_{14}),$$

whence $c_1 + c_6 = c_3 + c_4$. The above relations on the c_i yield $c_1 = c_2 = \cdots = c_6 = 0$, i.e., $D(x_i) \subset I_{i+1}$ for $i = 1, 2, \ldots, 6$. Hence $D^6(L) \subset [L, L]$. Since L is nilpotent, D is a nilpotent derivation.

REMARK. An analogous proof shows that every automorphism of L is unipotent.

PROPOSITION 2. *L is a derived algebra.*

PROOF. Let D_1 denote the derivation of L such that

$$\begin{aligned} &D_1(x_3) = x_7,\ D_1(x_4) = 2x_8,\ D_1(x_5) = 3x_9 + 2x_{11}, \\ &D_1(x_6) = 4x_{10} + 5x_{12},\quad D_1(x_8) = x_{15},\quad D_1(x_9) = 2x_{16}, \\ &D_1(x_{11}) = -x_{16},\quad \text{with } D_1(x_i) = 0 \text{ otherwise}. \end{aligned}$$

Let D_2 denote the derivation of L such that

$$\begin{aligned} &D_2(x_1) = x_2,\ D_2(x_2) = x_3,\ D_2(x_3) = x_4,\ D_2(x_4) = x_5, \\ &D_2(x_5) = x_6,\ D_2(x_6) = 0,\ D_2(x_7) = x_8,\ D_2(x_8) = x_9 + x_{11}, \\ &D_2(x_9) = x_{10} + x_{12},\ D_2(x_{10}) = x_{13} + x_{15},\ D_2(x_{11}) = x_{12}, \\ &D_2(x_{12}) = -x_{13} - (4/5)x_{15},\ D_2(x_{13}) = x_{14},\ D_2(x_{14}) = -x_{16}, \\ &D_2(x_{15}) = 0,\ D_2(x_{16}) = 0. \end{aligned}$$

One finds that $[D_1, D_2] = \operatorname{ad}(x_1)$. Hence we may extend L to a Lie algebra $H = (\tilde{D}_1, \tilde{D}_2) + L$ in which $[\tilde{D}_i, x_j] = D_i(x_j)$, $[\tilde{D}_1, \tilde{D}_2] = x_1$ and products in L are as before. Note that $[H, H] = L$.

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