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COMPACT COMPOSITION OPERATORS ON $B(D)$

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ABSTRACT. Let D be a domain in the complex plane, $\phi: D \to D$ be analytic, and $B(D)$ be the uniform algebra of bounded analytic functions on D with maximal ideal space M. The composition operator $C_\phi(f) = f \circ \phi$ is compact if and only if the weak* and norm closures of $\phi(D)$ coincide if and only if whenever the Euclidean closure of $\phi(D)$ contains a point λ of the boundary of D then each $f \in B(D)$ extends continuously from $\phi(D)$ to λ. If C_ϕ is compact, then either ϕ fixes a point of D or else the adjoint of C_ϕ fixes a point of M.

Introduction. Let D be a domain in the complex plane which supports nonconstant bounded analytic functions and let $B(D)$ be the uniform algebra of bounded analytic functions on D with supremum norm. Each analytic $\phi: D \to D$ defines the *composition operator* C_ϕ on $B(D)$ by $C_\phi(f) = f \circ \phi$ for all $f \in B(D)$. Each composition operator is clearly linear and norm reducing.

This paper consists of two parts. In §1 we characterize compact composition operators on $B(D)$, and in §2 we discuss fixed points of ϕ when C_ϕ is compact.

1. **Compact operators.** For each $z \in D$ denote by $\hat{z}$ the *evaluation functional* on $B(D)$ defined by $\hat{z}(f) = f(z)$ for each $f \in B(D)$. We can then consider D as a subset of $B(D)^*$. For each C_ϕ denote by $\Phi: B(D)^* \to B(D)^*$ the *adjoint* of C_ϕ defined by

$$\Phi(T)(f) = T(C_\phi(f)), \qquad f \in B(D),\ T \in B(D)^*,$$

so that if T is $\hat{z}$ for any $z \in D$ we have

$$\Phi(\hat{z})(f) = \hat{z}(C_\phi(f)) = \hat{z}(f \circ \phi) = f(\phi(z)) = (\phi(z))\hat{\ }(f),$$

and the function ϕ is the restriction of Φ to D.

We assume that each point λ in the boundary ∂D of D is essential for $B(D)$ in that there is some $f \in B(D)$ which does not extend to be analytic at λ. The domain D comes equipped with the usual topology from the plane induced by the chordal metric so that every closed subset of D is compact. D also inherits

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both the weak* and norm topologies from $B(D)^*$, and all three topologies agree inside D. For each subset A of D denote by $\operatorname{cl} A$ and w^* - $\operatorname{cl} A$ the Euclidean and weak* closures of A.

D is a subset of the maximal ideal space M of $B(D)$ but does not exhaust all of M. For each $\lambda \in \operatorname{cl} D$ the *fiber* M_λ over λ is the set of all $m \in M$ for which $m(f) = f(\lambda)$ whenever $f \in B(D)$ extends to be analytic at λ. Denote by B_1 the closed unit ball of $B(D)$. A linear operator L on $B(D)$ is called *compact* if $L(B_1)$ is relatively norm compact. Finally if $\{f_n\}$ is a sequence in $B(D)$ with $f_n \to f \in B(D)$ uniformly on compact subsets of D we write $f_n \to f$ ucc.

The following theorem of H. J. Schwartz [4, Theorem 2.5] can be proved by a simple normal families argument.

1.1. THEOREM. *A composition operator C_ϕ is compact on $B(D)$ if and only if for every sequence $\{f_n\}$ in B_1 with $f_n \to 0$ ucc we have $\|f_n \circ \phi\| \to 0$.*

Call a set A in M a *peak set* for $B(D)$ if there is some $f \in B_1$ whose Gel′ fand transform $\hat{f}$ is equal to 1 on A while $|\hat{f}(m)| < 1$ for all $m \in M - A$.

1.2. COROLLARY. *Let D be a domain for which the fiber M_λ is a peak set for $B(D)$ for every $\lambda \in \partial D$. Then C_ϕ is compact on $B(D)$ if and only if $\operatorname{cl} \phi(D)$ contains no point of ∂D.*

When there is a $\lambda \in \partial D$ whose fiber is a nonpeak set the situation is more complicated. T. W. Gamelin and J. Garnett [2] showed that if M_λ is not a peak set, then there is an unique $m_\lambda \in M_\lambda$ called the *distinguished homomorphism* with a representing measure living in $M - M_\lambda$. If we denote by $P(m_\lambda, \varepsilon)$ the open ε-ball about m_λ in the norm of $B(D)^*$, then $P(m_\lambda, \varepsilon) \cap D$ is nonempty for all $\varepsilon > 0$. Moreover $\{\lambda\}$ is a singleton component of ∂D.

We call a sequence $\{z_n\}$ in D an *interpolating sequence* if for every $\{s_n\} \in l^\infty$ there is some $f \in B(D)$ with $f(z_n) = s_n$ for all n. Interpolating sequences and distinguished homomorphisms are related by the following theorem [2, Theorem 3.5].

1.3. THEOREM. *If $\{z_n\}$ is a sequence in D which converges to some $\lambda \in \partial D$, then either $\{z_n\}$ contains an interpolating sequence or else M_λ is a nonpeak set, and $\{\hat{z}_n\}$ converges to the distinguished homomorphism m_λ in the norm of $B(D)^*$.*

1.4. COROLLARY. *The closure of D in the norm of $B(D)^*$ is the union of D and the set of distinguished homomorphisms.*

1.5. COROLLARY. *If C_ϕ is compact on $B(D)$, then $\phi(D)$ contains no interpolating sequences.*

Denote by Λ the set of distinguished homomorphisms, and for each $\varepsilon > 0$ define

$$K_\varepsilon = \text{w}^* \text{ - } \operatorname{cl} \phi(D) - \bigcup_{m_\lambda \in \Lambda} P(m_\lambda, \varepsilon).$$

1.6. THEOREM. *The following are equivalent*:

(a) *C_ϕ is compact on $B(D)$.*

(b) *The norm and weak* closures of $\phi(D)$ coincide.*

(c) *The only weak* cluster points of $\phi(D)$ in $M - D$ are distinguished homomorphisms.*

(d) *For every* $\varepsilon > 0$ *the set* K_ε *is a compact subset of* D.

PROOF. (a) *implies* (b). Let C_ϕ be compact. We show that every weak* cluster point of $\phi(D)$ is a norm cluster point. If $m \in M$ is a weak* cluster point of $\phi(D)$ there is a net $\{\phi(z_\alpha)\}$ converging weak* to m. The net $\{z_\alpha\}$ is an infinite subset of the weak* compact M and therefore has a subnet $\{z_\beta\}$ converging weak* to some $m_* \in M$. The net $\{\hat{z}_\beta\}$ is bounded and weak* convergent and C_ϕ is compact, so by [1, Theorem 6, p. 486], $\{\Phi(\hat{z}_\beta)\} = \{(\phi(z_\beta))^\wedge\}$ converges in norm $\Phi(m_*)$. At the same time $\{\phi(z_\beta)\}$ converges weak* to m so $\Phi(m_*) = m$.

(b) *implies* (c) *implies* (d). Trivial.

(d) *implies* (a). Let $\varepsilon > 0$. Without loss of generality we can assume ε so small that $K_{\varepsilon/8}$ is nonempty. Let $\{f_n\}$ be a sequence in B_1 with $f_n \to 0$ ucc. $K_{\varepsilon/8}$ is a nonempty compact subset of D, so there exists a natural number N such that $n \geqslant N$ implies $|f_n(z)| < \varepsilon/2$ for all $z \in K_{\varepsilon/8}$. Each $z \in \phi(D) - K_{\varepsilon/8}$ lies in some $P(m_\lambda, \varepsilon/8)$, and since $\phi(D)$ is connected the sets $K_{\varepsilon/8}$ and $P(m_\lambda, \varepsilon/4)$ must overlap. For any $z \in K_{\varepsilon/8} \cap P(m_\lambda, \varepsilon/4)$ and $w \in P(m_\lambda, \varepsilon/8)$ we have at the same time $|f_n(z)| < \varepsilon/2$ and $|f_n(z) - f_n(w)| < \varepsilon/2$ whenever $n \geqslant N$, so that $|f_n(w)| < \varepsilon$. The union of $K_{\varepsilon/8}$ and all the sets $P(m_\lambda, \varepsilon/8)$ covers $\phi(D)$, so we have $|f_n(z)| < \varepsilon$ for any $z \in \phi(D)$ whenever $n \geqslant N$, and therefore C_ϕ is compact by Theorem 1.1.

1.7. THEOREM. C_ϕ *is compact on* $B(D)$ *if and only if whenever the Euclidean closure of* $\phi(D)$ *contains a point* $\lambda \in \partial D$ *then* λ *possesses the distinguished homomorphism* m_λ *and each* $f \in B(D)$ *extends weak* continuously from* $\phi(D)$ *to* λ *according to* $f(\lambda) = m_\lambda(f)$.

PROOF. If C_ϕ is compact on $B(D)$ and cl $\phi(D)$ contains $\lambda \in \partial D$, then M_λ must be a nonpeak set with distinguished homomorphism m_λ. Let $\{z_n\}$ be a sequence in $\phi(D)$ with $z_n \to \lambda$. Corollary 1.5 says that $\{z_n\}$ cannot contain an interpolating sequence, so by Theorem 1.3 $\{\hat{z}_n\}$ converges in norm to m_λ. By part (b) of Theorem 1.6 the weak* closure of $\phi(D)$ contains no other points of M_λ and each $f \in B(D)$ extends weak*continuously from $\phi(D)$ to its weak* closure and therefore from $\phi(D)$ on λ according to $f(\lambda) = m_\lambda(f)$.

Conversely suppose C_ϕ is not compact. Then by part (c) of Theorem 1.6, $\phi(D)$ must have a weak* cluster point $m \in M - D$ which is not a distinguished homomorphism. Then $m \in M_\lambda$ for some $\lambda \in \partial D$. If λ does not possess a distinguished homomorphism, we are done. If there is $m_\lambda \in M_\lambda$ then $m \neq m_\lambda$.

If on the one hand m_λ is also a weak* cluster point of $\phi(D)$ there are nets $\{z_\alpha\}$ and $\{w_\beta\}$ in D with $\{\phi(z_\alpha)\}$ and $\{\phi(w_\beta)\}$ converging to m and m_λ respectively. Choose any $f \in B(D)$ with $\hat{f}(m) \neq \hat{f}(m_\lambda)$. Then this f has distinct weak* limits at λ.

If on the other hand m_λ is not a weak* cluster point of $\phi(D)$ then any $\{\phi(z_n)\}$ converging to λ contains an interpolating sequence $\{\phi(z_k)\}$ by Theorem 1.3, so there is an $f \in B(D)$ with $f(\phi(z_k)) = (-1)^k$, and this f is not continuous at λ.

We can now construct an example of a compact composition operator C_ϕ for which cl $\phi(D)$ contains a point of ∂D.

1.8. EXAMPLE. The earliest examples of domains with nonpeak fibers are the L-domains studied by L. Zalcman [5]. An *L-domain* is a domain obtained by excising from the punctured unit disc a sequence of disjoint closed discs $\Delta(x_n, r_n)$ whose centers $\{x_n\}$ are contained in the positive x-axis and accumulate only at 0. Zalcman showed that if $\sum r_n/x_n < \infty$, then M_0, the fiber over 0, is a nonpeak set, and the complex measure μ defined on ∂D by $d\mu = \zeta^{-1}\,d\zeta$ is finite and defines the distinguished homomorphism m_0 by

$$m_0(f) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta}\,d\zeta.$$

Then m_0 has a representing measure with no mass in M_0, and if Δ is a wedge in D centered on the negative x-axis with a vertex at 0, then Gamelin and Garnett showed [2, Theorem 5.1] that $\|\hat{z} - m_0\| \to 0$ as $z \to 0$ through Δ.

Let D be an L-domain for which M_0 is a nonpeak set and ϕ the restriction to D of a Riemann map from the disc to an open wedge in D with a vertex at 0. Then $0 \in \operatorname{cl}\phi(D)$, but C_ϕ is compact by Theorem 1.6 since each K_ε is a compact subset of D.

2. **Fixed points.** For each analytic $\phi\colon D \to D$ we define the *iterates* ϕ_n of ϕ by $\phi_0(z) = z, \ldots, \phi_{n+1}(z) = \phi(\phi_n(z)), \ldots$. If there is a point $w \in D$ such that $\phi(w) = w$ and $\phi_n(z) \to w$ for all $z \in D$ we call w an *attractive fixed point* of ϕ.

2.1. THEOREM. *If C_ϕ is compact on $B(D)$ then either ϕ has an attractive fixed point in D or else there is an unique $\lambda \in \partial D$ with distinguished homomorphism m_λ such that $\phi_n(z) \to \lambda$ for all $z \in D$, and $\Phi(m_\lambda) = m_\lambda$.*

PROOF. Suppose C_ϕ is compact and ϕ has no fixed point in D. We know ϕ cannot be a conformal automorphism of D, so according to a theorem of M. H. Heins [3, Theorem 2.2] there is a set A in ∂D with $\{\phi_n(z)\}$ converging to A in the sense that all the limit points of $\{\phi_n(z)\}$ are contained in A for all $z \in D$. A is either a singleton or a continuum. Since by Corollary 1.5 $\phi(D)$ contains no interpolating sequences A must contain only points with nonpeak fibers by Theorem 1.4. Each such point is a singleton component of ∂D, so there must be an unique $\lambda \in \partial D$ with distinguished homomorphism m_λ such that $\phi_n(z) \to \lambda$ for every $z \in D$. Furthermore $\{(\phi_n(z))\hat{}\,\}$ converges to m_λ in norm.

Now Φ is norm continuous so by Corollary 1.4 $\Phi(m_\lambda)$ must be either a point of D or a distinguished homomorphism. If $\Phi(m_\lambda)$ is a distinguished homomorphism it must be M_λ itself, and we are done.

If $\Phi(m_\lambda) = z_0 \in D$ we define the iterates of Φ in the same way we defined the iterates of ϕ, so that for $z \in D$ we have $\Phi_n(\hat{z}) = (\phi_n(z))\hat{}$. Then $\Phi_{n+1}(m_\lambda) = (\phi_n(z_0))\hat{}$, and $\{\Phi_{n+1}(m_\lambda)\}$ converges in norm to m_λ, but

$$\Phi_{n+1}(m_\lambda) = \Phi(\Phi_n(m_\lambda)) \to \Phi(m_\lambda)$$

in norm also, and we must have $\Phi(m_\lambda) = m_\lambda$ contradicting $\Phi(m_\lambda) = z_0$.

2.2. EXAMPLE. We show that there are functions ϕ without fixed points whose composition operators are compact. Let $\frac{1}{2} < r < 1$ and $\phi(z) = rz$. We construct an L-domain D so that $\phi(D) \subset D$ and C_ϕ is compact on $B(D)$, but ϕ fixes no point of D.

About r there is a closed disc $\Delta_1 = \Delta(r, \varepsilon_1)$ such that $\phi(\Delta_1)$ does not meet Δ_1. Inside $\phi(\Delta_1)$ there is a disc $\Delta(r^2, \varepsilon_2)$. Let $\Delta_2 = \Delta(r^2, \varepsilon_2/16)$. Then inside $\phi(\Delta_2)$

there is another disc $\Delta(r^3, \varepsilon_3)$. Let $\Delta_3 = \Delta(r^3, \varepsilon_3/64)$, and so on with $\Delta_n = \Delta(r^n, 2^{-2n}\varepsilon_n)$. Let D be the complement in the punctured disc of the union of the Δ_n's. Then $\phi(D) \subset D$, and cl $\phi(D)$ contains $0 \in \partial D$. Each $\varepsilon_n < 1$, and $\frac{1}{2} < r < 1$, so

$$\sum_{n=1}^{\infty} \frac{2^{-2n}\varepsilon_n}{r^n} \leqslant \sum \frac{1}{2^{2n} r^n} \leqslant \sum \frac{1}{2^n} < \infty$$

and M_0 is a nonpeak set by [5, p. 255].

Then the Cauchy integral formula [5, §4] produces a series expansion for f which can be shown to converge uniformly in $\phi(D) \cup \{0\}$ by imitating the proof of [5, Theorem 5.2], so that C_ϕ is compact by Theorem 1.7.

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