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ON hom dim $MU_*(X \times Y)$

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ABSTRACT. Let p be a prime and $B\mathbf{Z}/p$ the classifying space for the cyclic group $\mathbf{Z}/p$ of prime order p. A finite complex X is constructed such that

$$\hom \cdot \dim_{MU_*} MU_*(X \times B\mathbf{Z}/p) > \hom \cdot \dim_{MU_*} MU_*(X) + \hom \cdot \dim_{MU_*} MU_*(B\mathbf{Z}/p).$$

It has been widely expected that

$$\hom \cdot \dim_{MU_*} MU_*(X \times Y) \leqq \hom \cdot \dim_{MU_*} MU_*(X) + \hom \cdot \dim_{MU_*} MU_*(Y)$$

for X and Y CW complexes of finite type and $MU_*(\ )$ the complex bordism homology functor [2], [5, (6)]. Of particular interest has been the case $X = B\mathbf{Z}/p = Y$, where $B\mathbf{Z}/p$ is the classifying space of the cyclic group $\mathbf{Z}/p$ of prime order p, as in this case the inequality would imply an affirmative solution to a conjecture of Conner and Floyd [1, pp. 130–131]. The following is therefore something of a surprise.

THEOREM. *For each prime p there is a finite CW complex X with*

$$\hom \cdot \dim_{MU_*} MU_*(X) = 1$$

such that

$$\hom \cdot \dim_{MU_*} MU_*(X \times B\mathbf{Z}/p) \geqq 3 > 1 + 1 = \hom \cdot \dim_{MU_*} MU_*(X) + \hom \cdot \dim_{MU_*} MU_*(B\mathbf{Z}/p).$$

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To construct the relevant complex X we consider (for suitably large n) the pushout diagram

$$\begin{tikzcd} M(p; n + 2(p-1)) \arrow[r, "p^{p+1}"'] \arrow[d, "A"] & M(p^{p+2}; n + 2(p-1)) \arrow[d] \\ M(p, n) \arrow[r] & X \end{tikzcd}$$

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where $M(t; m) = S^m \cup_t e^{m+1}$, A is the map called α in [5] and Φ in [3], and p^{p+1} is the map of degree p^{p+1} on the bottom cell. There is thus a cofibration

$$M(p; n + 2(p-1)) \xrightarrow{A_* - p^{p+1}} M(p,n) \vee M(p^{p+2}; n + 2(p-1)) \to X$$

giving an exact triangle

$$\begin{tikzcd}[column sep=small]
MU_*(M(p,n+2(p-1))) \arrow[rr, "A_* - p^{p+1}"] & & MU_*(M(p,n)) \oplus MU_*(M(p^{p+2}; n+2(p-1))) \arrow[dl, "j_*"] \\
& MU_*(X) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

A moment's reflection shows that the horizontal map is monic, whence $\partial_* = 0$, and the resulting short exact sequence implies

PROPOSITION. *With the preceding notations, $MU_*(X)$ is generated by two classes, $u \in MU_n(X)$, $w \in MU_{n+2(p-1)}(X)$, satisfying the relations $pu = 0$, $[\mathbf{C}P(p-1)]u = p^{p+1}w$. Moreover* hom $\cdot$ $\dim_{MU_*} MU_*(X) = 1$.

PROOF. All that remains to be proved is the assertion about projective dimension. To this end note there is a commutative diagram

$$\begin{tikzcd}
MU_*(M(p;n)) \oplus MU_*(M(p^{p+2}; n+2(p-1))) \arrow[r] \arrow[d, "\mu", "\text{epic}"'] & MU_*(X) \arrow[r] \arrow[d, "\mu"] & 0 \\
H_*(M(p;n); \mathbf{Z}) \oplus H_*(M(p^{p+2}; n+2(p-1)); \mathbf{Z}) \arrow[r, "j_*"] & H_*(X; \mathbf{Z}) \arrow[dl, "\partial_*"] & \\
H_*(M(p, n+2(p-1))) \arrow[u, "A_* - p^{p+1}"] & &
\end{tikzcd}$$

Since $A_* = 0$ and p^{p+1} is monic, it follows that j_* is epic, whence the commutative square shows the Thom map μ: $MU_*(X) \to H_*(X; \mathbf{Z})$ is epic and the result follows from [2, 3.11]. $\square$

PROOF OF THEOREM. Recall [1, 46.3] that $MU_*(B\mathbf{Z}/p)$ is generated by classes $\alpha_{2k-1} \in MU_{2k-1}(B\mathbf{Z}/p)$ of additive order p^{a+1} where $2a(p-1) < 2k-1 < 2(a+1)(p-1)$ [1, 36.1]. There is (among many others!) the relation [1, p. 145(*)]

$$[V^{2p^2-2}]\alpha_1 + [\mathbf{C}P(p-1)]\alpha_{2p(p-1)+1} \in pMU_*(B\mathbf{Z}/p),$$

where $[V^{2p^2-2}]$ is a Milnor manifold of dimension $2p^2 - 2$. So write

$$[V^{2p^2-2}]\alpha_1 = px - [\mathbf{C}P(p-1)]\alpha_{2p(p-1)+1}.$$

From the Künneth exact sequence [2, 8.4], we see that

$$u \otimes \alpha_1 \neq 0 \in MU_*(X \times B\mathbf{Z}/p).$$

Note

$$\begin{aligned}
[V^{2p^2-2}]u \otimes \alpha_1 &= u \otimes [V^{2p^2-2}]\alpha_1 \\
&= u \otimes (px - [\mathbf{C}P(p-1)]\alpha_{2p(p-1)+1}) \\
&= pu \otimes x - [\mathbf{C}P(p-1)]u \otimes \alpha_{2p(p-1)+1} \\
&= 0 - p^{p+1} w \otimes \alpha_{2p(p-1)+1} \\
&= w \otimes p^{p+1}\alpha_{2p(p-1)+1} = u \otimes 0 = 0.
\end{aligned}$$

Therefore the annihilator ideal $A(u \otimes \alpha_1)$ contains $[V^{2p^2-2}]$. From degree considerations, $u \otimes \alpha_1$ is primitive; so, by the Ballantine lemma [3, II, 2.1], it follows that $A(u \otimes \alpha_1)$ also contains $[\mathbf{C}P(p-1)]$ and p. Hence,

$$\text{hom} \cdot \dim_{MU_*} MU_*(X \times B\mathbf{Z}/p) \geqq 3$$

by [3, 5.3]. $\square$

REMARK. By replacing $B\mathbf{Z}/p$ by a suitable large lens space $L(2m-1;p)$, we obtain *finite* complexes X, Y with MU hom $\cdot$ dim 1, whose Cartesian product has MU hom $\cdot$ dim at least 3.

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