PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 56, April 1976

DYNAMICAL SYSTEMS WITH CROSS-SECTIONS

DEAN A. NEUMANN

ABSTRACT. The problem of classifying dynamical systems (flows) with global cross-sections in terms of the associated diffeomorphisms of the cross-sections is considered. Suppose that, for $i = 1, 2$, ϕ_i is a C^r flow ($r \geqslant 0$) on the C^r manifold M_i that admits a global cross-section $S_i \subseteq M_i$ with associated diffeomorphism ('first return map') d_i. If rank $(H_1(M_1;\mathbf{Z})) = 1$, then (M_1, ϕ_1) is C^s equivalent ($s \leqslant r$) to (M_2, ϕ_2) if and only if d_1 is C^s conjugate to d_2. If rank $(H_1(M_1;\mathbf{Z})) \neq 1$ and ϕ_1 has a periodic orbit, then there are infinitely many global cross-sections $T_i \subseteq M_1$ of ϕ_1, such that the associated diffeomorphisms are pairwise nonconjugate.

1. **Introduction.** In this paper we consider the problem of the classification of dynamical systems (flows) with global cross-sections in terms of the associated diffeomorphisms of the cross-sections.

Suppose that S is a C^r manifold (connected, but not necessarily compact and possibly with nonempty boundary; $r \geqslant 1$) and that d is a C^r diffeomorphism of S. The *suspension* of d is a C^r flow ϕ: $M \times \mathbf{R}^1 \to M$ on an $(n+1)$-manifold M defined as follows: M is the quotient space of $S \times \mathbf{R}^1$ obtained by identifying each point (s,t) with $(d(s), t+1)$; ϕ is the flow on M induced by the constant vector field $(0, 1)$ on $S \times \mathbf{R}^1$ (cf. [8, §2]). We say that two diffeomorphisms, say (S,d) and (S',d'), are *flow equivalent* (C^s *flow equivalent*, $1 \leqslant s \leqslant r$) if the corresponding suspensions, (M,ϕ) and (M',ϕ'), are topologically equivalent (C^s equivalent) (i.e., if there is a homeomorphism (C^s diffeomorphism) h: $M \to M'$ that maps orbits of ϕ onto orbits of ϕ' and preserves the natural orientation of the orbits). It is known that if (M,ϕ) is a C^r flow that admits a global cross-section S, and d is the diffeomorphism of S induced by ϕ, then (M,ϕ) is topologically equivalent to the suspension of d [8, Theorem 2.2]. Also, if d and d' are topologically conjugate (C^s conjugate) (i.e., if $hd = d'h$ for some homeomorphism (C^s diffeomorphism) h: $S \to S'$), then (S,d) and (S',d') are (C^s) flow equivalent. We are interested in conditions under which the converse of the latter statement is true. Our main results are stated in Theorems 1 and 2 below.

THEOREM 1. *Suppose that d is a diffeomorphism of the C^r manifold S with suspension (M,ϕ) and that* rank $H_1(M) = 1$. *Then (S,d) is (C^s) flow equivalent to a diffeomorphism (S',d') if and only if d' is (C^s) topologically conjugate to d.*

Received by the editors July 25, 1975.
AMS (MOS) subject classifications (1970). Primary 58F99; Secondary 34C35, 54H20.

Here $H_1(M)$ denotes singular homology with integer coefficients. The *rank* of the abelian group A (not necessarily finitely generated) is defined to be the maximum number of elements of A/tA linearly independent over $\mathbf{Z}$; tA denotes the torsion subgroup of A.

We obtain as a corollary of Theorem 1 the following result of G. Ikegami [5]: "*If S is compact and there is no homomorphism of $\pi_1(S)$ onto the integers, then* (S, d) *and* (S', d') *are flow equivalent if and only if d and d' are topologically conjugate.*" We give examples in §3 to show that Theorem 1 is stronger than Ikegami's Theorem.

In fact the rank condition appears to be also necessary in order that a suspension (M, ϕ) admit an essentially unique section. We can prove this for a large class of diffeomorphisms, but not in general:

THEOREM 2. *Suppose that d is a diffeomorphism of the closed C^r-manifold S with suspension (M, ϕ) and that* rank $H_1(M) \neq 1$. *If d has at least one periodic point then there exist infinitely many pairwise nonconjugate diffeomorphisms (S_n, d_n), each flow equivalent to (S, d).*

REMARK. If the Euler characteristic of S is nonzero then any diffeomorphism of S has a periodic point [3]. Thus for such a manifold S, the suspension (M, ϕ) of a diffeomorphism d of S admits a unique (up to topological conjugacy) section if and only if rank $H_1(M) = 1$.

2. **Proof of Theorem 1**. Suppose that (M, ϕ) and (M', ϕ') are suspensions of (S, d) and (S', d') respectively, that h: $M \to M'$ is a topological equivalence of ϕ with ϕ', and that rank $H_1(M) = 1$. We will prove that d and d' are topologically conjugate.

$$\begin{tikzcd}
M \arrow[r, "h"] & M' \\
S \times \mathbf{R}^1 \arrow[u, "p"] \arrow[r, "\tilde{h}"] \arrow[d, "p_1"'] & S' \times \mathbf{R}^1 \arrow[u, "p'"'] \arrow[d, "p_1'"] \\
S \arrow[r, "\bar{h}"] & S'
\end{tikzcd}$$

First note that the natural projection p: $S \times \mathbf{R}^1 \to M$ is the projection of a regular covering space, with infinite cyclic group of covering transformations [7, Theorem 8.2, p. 165]. It follows that $\pi_1(M)$ decomposes as a semidirect product $\pi_1(S) \cdot \mathbf{Z}$ (the image $p_* \pi_1(S)$ is a normal subgroup of $\pi_1(M)$ with quotient $\mathbf{Z}$); similarly $\pi_1(M') = \pi_1(S') \cdot \mathbf{Z}$. Hence the commutator subgroup of $\pi_1(M)$ is contained in the subgroup $\pi_1(S)$ (i.e., $p_* \pi_1(S)$). Thus a generator z of the infinite cyclic factor of $\pi_1(M)$ goes onto the generator of an infinite cyclic *direct summand* of $H_1(M)$ under the abelianizing (Hurewicz) homomorphism $\alpha: \pi_1(M) \to H_1(M)$, and $\pi_1(S)$ is mapped by α onto the complementary summand. Since rank $H_1(M) = 1$ the complementary summand is the torsion subgroup; it follows that $\pi_1(S)$ consists of just those elements of $\pi_1(M)$ that are torsion modulo the commutator subgroup. This is also true of $\pi_1(S') \subseteq \pi_1(M')$ and, as this subgroup is characteristic, we see that $h_*: \pi_1(M) \to \pi_1(M')$ maps $\pi_1(S)$ isomorphically onto $\pi_1(S')$. We can now apply the *lifting criterion* to obtain a homeomorphism $\tilde{h}: S \times \mathbf{R}^1 \to S' \times \mathbf{R}^1$ that makes the top rectangle in the accompanying diagram commute. Since h maps orbits of ϕ onto orbits of ϕ', it follows that $\tilde{h}$ maps 'vertical' lines $\{s\} \times \mathbf{R}^1$ onto vertical lines. Thus $\tilde{h}$ induces a homeomorphism $\bar{h}: S \longrightarrow S'$ that makes the bottom rectangle in the diagram commute.

In fact $\bar{h}$ is a conjugacy of d with d'. To see this let τ be the covering transformation of $S \times \mathbf{R}^1$ defined by $\tau(s,t) = (d(s), t+1)$, and let τ' on $S' \times \mathbf{R}^1$ be defined analogously. Then

$$\bar{h}d(s) = p_1' \tilde{h}(d(s), 1) = p_1' \tilde{h}\tau(s,0) = p_1' \tau' \tilde{h}(s,0);$$

but $\tilde{h}(s,0) \in (p_1')^{-1}(\bar{h}(s))$, so $\tau'\tilde{h}(s,0) \in (p_1')^{-1}(d'\bar{h}(s))$; that is, $p_1'\tau'\tilde{h}(s,0) = d'\bar{h}(s)$ as required.

If the equivalence $h: M \to M'$ is a C^s diffeomorphism $(1 \leqslant s \leqslant r)$, then $\tilde{h}$ is a C^s diffeomorphism and so is $\bar{h}$; i.e., under the rank assumption, (S, d) and (S', d') are C^s flow equivalent if and only if d and d' are C^s conjugate.

REMARK. Theorem 1 may be considerably extended in the case of continuous flows. Let X be a connected, locally arcwise connected topological space and let f be a homeomorphism of X. There is a continuous flow $\tilde{\phi}$ on $X \times \mathbf{R}^1$ defined by $\tilde{\phi}(x,s,t) = (x, s+t)$, and $\tilde{\phi}$ induces a continuous flow ϕ on the quotient space Q obtained by identifying each point $(x,s) \in X \times \mathbf{R}^1$ with $(f(x), s+1)$. The argument given above carries over verbatim to prove

THEOREM 1′. *Suppose that f and f' are homeomorphisms of connected locally arcwise connected spaces X and X', with corresponding suspensions (Q, ϕ) and (Q', ϕ'). Assume that* rank $H_1(Q) = 1$. *Then ϕ is topologically equivalent to ϕ' if and only if f is topologically conjugate to f'.*

3. **Ikegami's Theorem.** An immediate consequence of Theorem 1 is the following result of Ikegami [5], [6]:

COROLLARY. *Suppose that S is a closed C^r n-manifold and that $\pi_1(S)$ admits no homomorphism onto the integers. Then (S, d) and (S', d') are flow equivalent if and only if d and d' are topologically conjugate.*

EXAMPLES. We give some examples to show that Theorem 1 is stronger than Ikegami's Theorem. In place of S we take the n-dimensional torus T^n; let d be a diffeomorphism of T^n. Then d_* is an automorphism of the free abelian group $\pi_1(T^n) \cong \mathbf{Z}^n$ and hence may be represented by a matrix $A \in \mathrm{GL}_n(\mathbf{Z})$. (Note that any $A \in \mathrm{GL}_n(\mathbf{Z})$ may be obtained in this way: if l is the linear homeomorphism of $\mathbf{R}^n$ represented by A, then l induces a diffeomorphism d of T^n, and we may choose a basis for $\pi_1(T^n)$ with respect to which d_* is represented by A.)

In the semidirect product $\pi_1(M) = \pi_1(T^n) \cdot \mathbf{Z}$, the action on $\pi_1(T^n)$ of an (appropriately chosen) generator z of the infinite cyclic factor is given by:

$z^{-1}sz = d_*(s)$ $(s \in \pi_1(T^n))$. Hence $\pi_1(M)$ has presentation

$$\langle e_1, \ldots, e_n, z \| [e_i, e_j] = 1; z^{-1}e_i z = d_*(e_i); i,j = 1, \ldots, n \rangle,$$

where $\{e_1, \ldots, e_n\}$ is a basis for $\pi_1(T^n)$ and $[e_i, e_j]$ denotes the commutator of e_i and e_j. It follows that $H_1(M) = B \oplus \mathbf{Z}$, where B has presentation

$$\langle e_1, \ldots, e_n \| [e_i, e_j] = 1, e_i^{-1} d_*(e_i) = 1, i,j = 1, \ldots, n \rangle.$$

We see from this that $A - I$ is a *presentation matrix* of B (i.e., there is a free resolution $0 \to \mathbf{Z}^n \xrightarrow{i} \mathbf{Z}^n \to B \to 0$ of B, with i represented by the matrix $A - I$) and that B is completely determined by the invariant factors of $A - I$. B is a torsion group if and only if no invariant factor of $A - I$ is zero, i.e., if and only if 1 is not an eigenvalue of A. Thus we have proved:

Suppose d is a diffeomorphism of T^n such that d_* does not have 1 *as an eigenvalue. Then a diffeomorphism d' of T^n is flow equivalent to d if and only if it is topologically conjugate to d.*

Of course, Ikegami's Theorem does not apply to any of these examples.

4. **Proof of Theorem 2.** Assume that d is a diffeomorphism of the closed C^r manifold S with suspension (M, ϕ) and that rank $H_1(M) \geqslant 2$. To find a section $S_n \subseteq M$ for ϕ distinct from S, we construct a C^r map P: $M \to S^1 \times S^1$; S_n will be realized as the inverse image of a submanifold $W_n \subseteq S^1 \times S^1$ on which P is transverse regular.

We first show that there is an epimorphism π: $\pi_1(S) \to \mathbf{Z}$ satisfying $\pi d_*^{-1} = \pi$. Since $\pi_1(M) \cong \pi_1(S) \cdot \mathbf{Z}$ we see that $H_1(M) \cong B \oplus \mathbf{Z}$ and that the Hurewicz homomorphism α: $\pi_1(S) \to H_1(M)$ maps $\pi_1(S)$ onto B. By computing $\pi_1(M)$ from Van Kampen's Theorem, we may check that a generator z of the infinite cyclic factor can be chosen so that its action on $\pi_1(S)$ is given by: $z^{-1}\sigma z = d_*^{-1}(\sigma)$ $(\sigma \in \pi_1(s))$. Hence for any $\sigma \in \pi_1(S)$,

$$\alpha d_*^{-1}(\sigma) = \alpha(z^{-1}\sigma z) = \alpha(\sigma).$$

Since rank $H_1(M) \geqslant 2$, there is an epimorphism β: $B \to \mathbf{Z}$; we may take $\pi = \beta \circ \alpha|_{\pi_1(S)}$.

We may now construct P as follows. There is a C^r map p: $S \to S^1$ with $p_* = \pi$. Since $p_* = p_* d_*^{-1}$ we see that there is a homotopy p_t: $S \to S^1$ $(t \in I = [0, 1])$ of p to pd^{-1}. By [4, Lemma 2] we may assume that $\{p_t\}$ defines a C^r map of $S \times I$ onto S^1 and that $p_t = p$ for $t \in [0, 1/3]$ and $p_t = pd^{-1}$ for $t \in [2/3, 1]$. Then the map $(s, t) \to (p_t(s), t)$ of $S \times I$ onto $S^1 \times I$ is compatible with the identifications $(s, 0) \leftrightarrow (d(s), 1)$ on $S \times I$ and $(s, 0) \leftrightarrow (s, 1)$ on $S^1 \times I$, and hence induces a C^r map P: $M \to S^1 \times S^1$.

We now want to determine a condition on a 1-dimensional submanifold $W \subseteq S^1 \times S^1 = T^2$ in order that P be transverse regular on W. On both M and T we consider only local coordinates that respect the "product" structure; viz., on M we choose coordinates of the form (x, t), where $x = (x_1, \ldots, x_n)$ are local coordinates on S and $t \in J$ (open) $\subseteq I$ (corresponding to a product chart on $S \times \mathbf{R}^1$), and on T^2 coordinates of the form (s, t), $t \in J$ (open) $\subseteq I$. With respect to the corresponding bases $\{\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial t\}$ of the tangent space $T_{(x,t)}M$ and $\{\partial/\partial s, \partial/\partial t\}$ of $T_{P(x,t)}T^2$, the derivative dP of P is represented locally by the matrix

$$\begin{bmatrix} \dfrac{\partial p_t}{\partial x_1} & \cdots & \dfrac{\partial p_t}{\partial x_n} & \dfrac{\partial p_t}{\partial t} \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

so that the tangent vector $(0, \ldots, 0, 1)$ (i.e., $\partial/\partial t$) in $T_{(x,t)}M$ is taken onto $(\partial p_t/\partial t, 1)$ in $T_{P(x,t)}T^2$. If we fix finite covers of M and T^2 consisting of charts of the above form, then

$$m = \sup|\partial p_t/\partial t| < \infty$$

(the supremum taken over all local representations of P with respect to these fixed charts). The condition that P be transverse regular on $W \subseteq T^2$ at $w \in W$ is that for any $(x,t) \in P^{-1}(w)$ we have $(\partial p_t/\partial t, 1) \notin T_w W$. Thus P will be transverse regular on W if, for all $w \in W$ and $(a,b) \in T_w W$, we have $|b/a| > m$. It is clear that, for all sufficiently large integers k, there is a C^r simple closed curve $W_k \subseteq T^2$ that satisfies this condition and winds k-times around the $S^1 \times \{0\}$ factor of T^2 and once around the $\{s_0\} \times S^1$ factor.

Now fix such a submanifold $W_k \subseteq T^2$ and let $S_k \subseteq M$ denote $P^{-1}(W_k)$. Since P is transverse regular on W_k, S_k is a codimension one C^r-submanifold of M that is transverse to the flow ϕ [1, Theorem 17.1]. It is also true that S_k is connected. We may see this as follows: T^2 fibers over S^1 with fibers simple closed curves "parallel" to W_k; let q: $T^2 \to S^1$ denote the projection of such a fibering and let $Q = q \circ P$. We may assume that P is transverse regular on each fiber of q, and hence that each point of S^1 is a regular value of Q. It follows that Q fibers M over S^1 with each fiber diffeomorphic to S_k (cf. [2, §1.1]). Since $q_*: \pi_1(T^2) \to \pi_1(S^1)$ and $P_*: \pi_1(M) \to \pi_1(T^2)$ are surjective, so is Q_*. Because M is connected and Q_* is surjective, we see that S_k is connected.

To prove that S_k is a global section for ϕ we must show that for any $s \in S_k$ there is a time $\tau > 0$ with $\phi(s, \tau) \in S_k$ [8, §2]. We will need slightly more than this to see that the diffeomorphism d_k, induced on S_k by ϕ, is not conjugate to d; viz., that, for all sufficiently large k, each orbit of ϕ crosses S_k at least some fixed number $j \geqslant 2$-times between successive crossings of S. But if k is large enough then W_k meets each "vertical" segment $s' \times [0, 1/3]$ at least j-times in $S^1 \times [0, 1/3]$. It follows that each orbit segment $s \cdot [0, 1/3] = \{\phi(s,t) | t \in [0, 1/3]\}$, with $s \in S$, meets S_k at least j-times, as asserted.

We now make use of our assumption that d has at least one periodic orbit. Let m (m_k) be the minimal period of periodic orbits of d (d_k). We have proved that $m_k \geqslant j \cdot m$, and hence that (S, d) and (S_k, d_k) are not topologically conjugate.

BIBLIOGRAPHY

1. R. Abraham and J. Robbin, *Transversal mappings and flows*, Benjamin, New York and Amsterdam, 1967. MR **39** #2181.

2. W. Browder and J. Levine, *Fibering manifolds over a circle*, Comment.Math. Helv. **40** (1966), 153–160. MR **33** #3309.

3. F. B. Fuller, *The existence of periodic points*, Ann. of Math. (2) **57** (1953), 229–230. MR **14**, 669.

4. S. T. Hu, *On singular homology in differentiable spaces*, Ann. of Math. (2) **50** (1949), 266–269. MR **10**, 728.

5. G. Ikegami, *On classification of dynamical systems with cross-sections*, Osaka J. Math. **6** (1969), 419–433. MR **42** #1131.

6. ———, *Flow equivalence of diffeomorphisms*. I, II, Osaka J. Math. **8** (1971), 49–69, 71–76. MR **44** #4780.

7. W. S. Massey, *Algebraic topology: An introduction*, Harcourt, Brace & World, New York, 1967. MR **35** #2271.

8. S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 97–116. MR **29** #2818b.

Department of Mathematics, Bowling Green State University, Bowling Green, Ohio 43403