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CORRIGENDUM AND ADDENDUM TO "PSEUDO-MATCHINGS OF A BIPARTITE GRAPH"

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ABSTRACT. For an undirected bipartite graph G conditions on $\deg x + \deg y$ for $xy \notin G$ ensure sets of independent lines extend to hamiltonian cycles.

Let X, Y be two finite sets each with $n \geqslant 2$ elements. Also let G be an undirected bipartite graph with points $X \cup Y$ and lines xy with $x \in X, y \in Y$. We will assume $\deg x + \deg y \geqslant \delta$ whenever $xy \notin G$. By HC and HP we mean hamiltonian cycle and path respectively. Consider the following:

THEOREM 1. *Every set of independent lines of G extends to a matching iff each connected component of G is a complete graph $K_{m,m}$.*

THEOREM 2. *If* $\deg v > \frac{1}{2}n$ *for each point v then G has an HC.*

THEOREM 3. (i) *If $\delta = l + m$ where $1 \leqslant l \leqslant m \leqslant n$ then every set of $\leqslant l$ independent lines of G extends to a set of m independent lines.*

(ii) *If $\delta = n + 1$ every set of independent lines of G extends to a matching and every matching extends to an HP. Also some $n - 1$ lines of any matching extend to an HC.*

(iii) *If $\delta = n + 2$ every matching of G extends to an HC.*

(iv) *If $\delta = \frac{1}{3}(4n + 1)$ any set of independent lines of G extends to an HC.*

(v) *If $\delta = n + 1 + \frac{1}{2}k$ then any path of length k in G extends to an HC.*

The proof of Theorem 1 is easy. Theorem 2 appeared in [3] but this writer introduced an error into the proof by claiming that $n + 2$ can be replaced by $n + 1$ in Theorem 3(iii). Once this is realised the necessary corrections for [3] are readily made. In fact a stronger form of Theorem 2 appeared in [6]. Clearly Theorem 3(ii) is stronger than Theorem 2. That G has an HC when $\delta = n + 1$ was proved in [2]. It is interesting to compare the results here with those in [4]. Theorem 3(i) follows from the result about the largest set of independent lines in G (see [1, Theorem 4, p. 98]). We will need a result [7, Theorem 2D] of D. R. Woodall, whose kind suggestions improved an earlier version of this note, namely

THEOREM 4. *Let F be a directed graph on n vertices with* out $\deg u$ + in $\deg v \geqslant \Delta$ *whenever $u \to v$ is not an arc of F.*

(i) *If $\Delta = n - 1$ then F has a directed HP.*

(ii) *If $\Delta = n$ then F has a directed HC.*

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Notice that Theorem 4(i) follows from Theorem 4(ii) by adding a new point w to F and all possible arcs between w and F. In the same way Meyniel's Theorem [5] shows that if

$$\text{out deg } u + \text{in deg } u + \text{out deg } v + \text{in deg } v \geqslant 2n - 3$$

whenever $u \to v$ and $v \to u$ are not arcs of F then F has a directed HP, but we will not need this stronger result.

PROOF OF THEOREM 3(ii) AND 3(iii). By Theorem 3(i) there is a matching; let X, Y be numbered so it is $x_i y_i$ for $1 \leqslant i \leqslant n$. Draw a directed graph F with vertices $z_1, \ldots, z_n$ in which $z_i \to z_j$ if $i \neq j$ and $x_i y_j \in G$. Then F has $\Delta = \delta - 2$. By Theorem 4 there is an HP or an HC in F which yields an HP or an HC as the case may be of G containing the given matching. For the last part of Theorem 3(ii) we suppose the HP is $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. We suppose $x_1 y_n \notin G$ for otherwise we have the desired HC. Then $\deg x_1 + \deg y_n \geqslant n + 1$ and there is an i in $1 < i < n$ with $x_1 y_i$ and $x_i y_n$ both in G, so

$$x_1, y_1, x_2, y_2, \ldots, x_i, y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_{i+1}, x_{i+1}, y_i, x_1$$

is the desired HC. To see that 3(iii) is best possible consider two complete bipartite graphs with exactly one edge in common and take any matching which contains this edge.

PROOF OF THEOREM 3(iv). Suppose $n \geqslant 3$ and there is a graph G, with the maximum possible number of lines, such that some set L of independent lines does not extend to an HC. Then there is a line xy not in G and L extends to an HC in $G + xy$. Let the HC be

$$x = x_1, y_1, x_2, y_2, \ldots, x_n, y_n = y, x_1 .$$

Of course adjacent lines of this cycle are not both in L. Let I be the set of integers i such that xy_i and $x_i y$ are both in G. Then $1, n \notin I$ and if $k = |I|$ we have $\frac{1}{3}(4n + 1) \leqslant \deg x + \deg y \leqslant n + k$ so $k \geqslant 2$. Also $x_i y_i \in L$ for $i \in I$ as otherwise

$$x_1, y_1, x_2, y_2, \ldots, x_{i-1}, y_{i-1}, x_i, y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_{i+1}, x_{i+1}, y_i, x_1$$

is an HC containing L. Next notice that if s is the smallest integer in I then $y_{s-1} x_{i+1} \notin G$ for any given $i \in I$, for otherwise

$$x_1, y_1, x_2, y_2, \ldots, x_{s-1}, y_{s-1}, x_{i+1}, y_{i+1}, x_{i+2}, y_{i+2}, \\ \ldots, x_n, y_n, x_s, y_s, \ldots, x_i, y_i, x_1$$

is an HC containing L. It follows that $\deg y_{s-1} \leqslant n - k$ and by symmetry $\deg x_{r+1} \leqslant n - k$ where r is the largest integer in I. Hence $2(n - k) \geqslant \deg y_{s-1} + \deg x_{r+1} \geqslant \frac{1}{3}(4n + 1)$ which contradicts the earlier result $\frac{1}{3}(4n + 1) \leqslant n + k$.

Let $n = 3m + \lambda$ where $\lambda \in \{-1, 0, 1\}$, and $W \subset X$ with $|W| = m$, and $Z \subset Y$ with $|Z| = 2m - 1$. Let G be the graph containing all lines except xy with $x \in W, y \notin Z$. Then any matching of Z into $X \backslash W$ does not extend to an HC. This example has $\deg x + \deg y = [\frac{1}{3}(4n + 1)] - 1$ and so shows

Theorem 3(iv) is best possible when $\lambda = -1$. Probably $\frac{1}{3}(4n + 1)$ can be replaced by its integral part for $n \geqslant 5$, but it seems to be hard to obtain this slight improvement.

Notice that if $\delta = n + |L| \geqslant n + 2$ then L extends by (i) to a matching, which in turn extends to an HC by (iii), but is this result best possible?

PROOF OF THEOREM 3(v). Suppose there is a graph G, with the maximum possible number of lines, such that some path P of even length k does not extend to an HC. Then there is a line xy not in G and P extends to an HC in $G + xy$. The value of δ ensures we can choose an edge $x'y'$ in this HC but not in P, such that xy', $x'y$ both lie in G. Hence we get a contradictory HC containing P. Examples with G consisting of two overlapping complete bipartite graphs and P filling the intersection, show the result best possible for $k \leqslant 2n - 4$. For $k \geqslant 2n - 3$ by inspection the graph must be complete if P is to extend to an HC.

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