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WHEN IS $D + M$ COHERENT?

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ABSTRACT. Let V be a valuation ring of the form $K + M$, where K is a field and $M(\neq 0)$ is the maximal ideal of V. Let D be a proper subring of K. Necessary and sufficient conditions are given that the ring $D + M$ be coherent. The condition that a given ideal of V be $D + M$-flat is also characterized.

1. **Introduction and notation.** Let V be a valuation ring of the form $K + M$, where K is a field and M $(\neq 0)$ is the maximal ideal of V. Let D be a proper subring of K; let k, viewed inside K, be the quotient field of D. Our purpose is twofold: to answer the question raised in the title, and to determine when a given ideal of V is a flat $D + M$-module. Entering into the solution is a result of Ferrand [6] on the descent of flatness.

It is well known (cf. [7, Theorem A(m), p. 562]) that $D + M$ is Noetherian (and, hence, coherent) if and only if the following hold: $D = k$, K is a finite (algebraic) extension of k, and V is discrete rank one. One upshot of considering coherence for $D + M$ is the relaxing of the first and third of these conditions. (See Theorem 3 and Remark 6(a) below.) To motivate the second problem, note that the riding homological assumption used by Greenberg and Vasconcelos [10] to study coherence for a certain family of pullbacks is, when specialized to the $D + M$-construction, the condition that ($k = K$ and) M is $D + M$-flat. Inasmuch as V is coherent and M is V-flat, it is striking (see Corollary 8) that, in case $D = k$, the coherence of $D + M$ forces M to be nonflat over $D + M$.

Background material on the $D + M$-construction and coherence may be found in [7, Appendix 2] and [2], respectively. In addition to the notation fixed above, it will be convenient to denote $D + M$ by R.

2. **Coherence and flatness.** Before answering the question raised in the title, we give two lemmas.

LEMMA 1. *Let M be a finitely generated ideal of R. Then $D = k$.*

PROOF. Since $M \neq 0$, it follows from Nakayama's lemma that $M \neq M^2$. Now M is cyclic as a V-module (since V is valuation), so that M/M^2 is cyclic over $V/M = K$. Thus M/M^2 and K are isomorphic as K-spaces and, *a fortiori*,

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as D-modules. However, M/M^2 is finitely generated over $R/M = D$, so that K is a finitely generated D-module. By integrality [1, Lemma 2, p. 326], D is a field, as required.

An integral domain T, with quotient field L, is said to be *finite-conductor* in case $Ta \cap Tb$ is a finitely generated T-module for each a, b in L. (Finite-conductor domains have figured recently in [11], [3], and [4].) By [2, Theorem 2.2], any coherent domain is finite-conductor.

LEMMA 2. *If R is finite-conductor and $k \neq K$, then M is a finitely generated ideal of R and $D = k$.*

PROOF. Select b in $K \backslash k$ and nonzero m in M. We claim that $Rm \cap Rbm = Mm$. Indeed, containment one way is clear, as $M = Mb$. Conversely, if r is in $Rm \cap Rbm$, then

$$r = (d_1 + m_1)m = (d_2 + m_2)bm$$

for some d_i in D, m_i in M $(i = 1,2)$. Cancellation gives $d_1 + m_1 = d_2 b + m_2 b$ and so $d_1 = d_2 b$. Since b is not in k, we have $d_2 = 0 = d_1$, so that $r = m_1 m$ is in Mm, thus sustaining the claim. As R is finite-conductor, Mm is a finitely generated ideal of R. However, Mm and M are R-isomorphic, and an application of Lemma 1 completes the proof.

THEOREM 3. *R is coherent if and only if one of the following conditions holds:*
(1) *$k = K$ and D is coherent;*
(2) *M is a finitely generated ideal of R.*
Moreover, if condition (2) *holds, then $D = k$ and K is a finite extension of k.*

PROOF. Assume that R is coherent. If $k = K$, an easy direct argument or an appeal to [8, Theorem 5.14] shows that D is coherent. Now suppose that $k \neq K$. By Lemma 2, M is finitely generated over R.

Conversely, we see directly or by [8, Theorem 5.14] that condition (1) implies that R is coherent. Next, assume that (2) holds. By the proof of Lemma 1, $D = k$ and there is an integer $n \geqslant 2$ such that $K \cong M/M^2 \cong k^n$ isomorphic as k-spaces. (This yields the final assertion of the theorem.) In particular, V is a finitely generated R-module.

We claim that V is finitely presented over R. Let $\{b_i\colon 1 \leqslant i \leqslant n\}$ be any k-basis of K. If R^n is R-free on a basis $\{e_i\colon 1 \leqslant i \leqslant n\}$, the R-module homomorphism $g\colon R^n \to V$ determined by $g(e_i) = b_i$ is surjective. It is straightforward to verify that the R-module $\ker(g)$ is isomorphic to the direct sum of $n - 1$ copies of M. By (2), $\ker(g)$ is finitely generated over R, thus establishing the above claim.

To show that R is coherent, we use the criterion in [2, Theorem 2.1(a)]; viz., we show that the product of any family $\{A_j\colon j \text{ in } J\}$ of flat R-modules is flat. As each $A_j \otimes_R V$ is V-flat and V is coherent, $\Pi(A_j \otimes_R V)$ is V-flat. However, since V is finitely presented over R, the canonical homomorphism $(\Pi A_j) \otimes_R V \to \Pi(A_j \otimes_R V)$ is an isomorphism (cf. [1, Exercise 9(a), p. 43]), so that $(\Pi A_j) \otimes_R V$ is also V-flat. That ΠA_j is R-flat, now follows from Ferrand's descent result [6, Lemme], as applied to the inclusion $R \to V$, and completes the proof.

In view of [11, Theorem 1], the next result may be used to recover [7, Theorem A(i), p. 561].

COROLLARY 4. *R is integrally closed and coherent if and only if $k = K$ and D is integrally closed and coherent.*

PROOF. Combine Theorem 3 with [7, Theorem A(b), p. 560] and [8, Theorem 5.14].

We next focus on condition (2) of Theorem 3, in order to prepare for the examples below.

COROLLARY 5. *Let $D = k$. Then R is coherent if and only if K is a finite extension of k and $M \neq M^2$.*

PROOF. By [9, Lemma 1.3], $M \neq M^2 \Leftrightarrow M$ is a principal ideal of V. The "only if" half is now immediate from Theorem 3. Conversely, if $\{b_i\colon 1 \leqslant i \leqslant n\}$ is a (finite) k-basis of K and $M = Vm$ for some m in M, then $\{b_i m\colon 1 \leqslant i \leqslant n\}$ is easily seen to generate M as an R-module, and an application of Theorem 3 completes the proof.

REMARK 6. (a) One may ask whether $k \neq K$ and R coherent imply R Noetherian. By Theorem 3 and Corollary 5 (and the result quoted in the introduction), the answer is affirmative if and only if V has rank one.

To construct an instance where the answer is negative, let K/k be a nontrivial finite field extension. As in [1, Example 6, p. 390], construct a valuation ring $V = K + M$ whose corresponding valuation v has (rank two) value group $\mathbf{Z} \times \mathbf{Z}$, with the lexicographic order. As M is the set of elements b in the quotient field of $K + M$ such that $v(b) > 0$, every element d in M^2 satisfies $v(d) \geqslant (0,2)$. Select e with $v(e) = (0,1)$; then e is in $M \backslash M^2$ and, by Corollary 5, $R = k + M$ is the desired example.

(b) The condition "$M \neq M^2$" in Corollary 5 cannot be deleted. Indeed, let K/k again be nontrivial finite, with $V = K + M$ having value group $\mathbf{R}$. Since $\mathbf{R} = 2\mathbf{R}$, it is clear that $M = M^2$, so that $R = k + M$ is not coherent. (To produce an example with V of rank exceeding one, traffic similarly with the lexicographically ordered value group $\mathbf{R} \times \mathbf{R}$.)

THEOREM 7. *Let I be a nonzero ideal of V. Then I is R-flat if and only if at least one of the following conditions holds*:

(1) *$k = K$;*

(2) *I is not a principal ideal of V.*

NOTE. If condition (1) holds, then I/MI is D-flat; by [9, Lemma 1.3], $(2) \leftrightarrow I = MI$.

PROOF. Suppose that $k = K$. Then, $V = R_M$ is R-flat; moreover, I is V-flat, since any ideal of V is. Thus, transitivity of flatness shows that I is R-flat, whence $I/MI \cong I \otimes_R D$ is D-flat.

Next, suppose that I is R-flat and $k \neq K$. If (2) fails, then $I \neq MI$. Now $I \otimes_R (k + M) = I$ is a flat ideal of $k + M$, so that [12, Lemma 2.1] implies that I is a principal ideal of $k + M$. Then $k \cong I/MI \cong K$, contradicting $k \neq K$. This concludes the "only if" part of the proof.

It remains to show that condition (2) guarantees that I is R-flat. Let a, b be elements of I; without loss of generality, a divides b in V. By (2), $I = MI$, so that $a = \sum e_i d_i$, for some e_i in M, d_i in I. Without loss of generality, $e = e_1$ divides e_i for each $i > 1$, so that a and b are each in $Ie \subset Re$. Thus, I is the (filtered) direct limit of its principal subideals over R, hence is flat, to complete the proof.

Theorem 7 and Corollary 5 will be used by one of us in a subsequent paper in order to answer a question raised in [4] about rings of global dimension 3. Combining Theorem 3 with Theorem 7 (for the case $I = M$) leads immediately to our next result.

COROLLARY 8. *R is coherent and M is R-flat if and only if $k = K$ and D is coherent.*

We close with a homological remark.

REMARK 9. Let R be coherent, such that $k \neq K$. Then M has infinite projective dimension over R. For a proof, $D = k$ by Theorem 3, so that [5, Corollary] implies that R is a going-down ring. If the result is denied, [4, Proposition 2.5] shows R is Prüfer, and [7, Theorem A(i), p. 561] then yields $k = K$, the desired contradiction.

In view of Theorem 3 and Corollary 5, the next result generalizes the assertion in the preceding paragraph (and has a more straightforward proof). If $M \neq M^2$ and $k \neq K$, then M has infinite flat (weak) dimension over R. For a proof, we may take $D = k$ since $M \otimes_R (k + M) = M$. By Theorem 7, M is not R-flat, and so the proof of [4, Proposition 4.5] may be modified to give the desired result.

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