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CESÀRO SUMMABILITY OF THE CONJUGATE SERIES AND THE DOUBLE HILBERT TRANSFORM

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ABSTRACT. If $f(x, y)$, a 2π periodic function in each variable, has a modulus of continuity $w_f(\delta) = o(1/\log(1/\delta))$ then

$$\tilde{\sigma}_n(x,y,f) - \int_{1/n}^{\pi}\int_{1/n}^{\pi} \frac{[f(x+u,y+v) - f(x-u,y+v) - f(x+u,y-v) + f(x-u,y-v)]}{4\tan(u/2)\tan(v/2)}\,du\,dv \to 0 \quad \text{uniformly in } (x,y)$$

where $\tilde{\sigma}_n(x, y, f)$ is the first arithmetic mean of the conjugate series. This theorem is best possible in that $o(1/\log(1/\delta))$ cannot be replaced by $O(1/\log(1/\delta))$.

Given a 2π periodic function $f(x, y)$ we shall define $\tilde{f}(x, y)$, the conjugate of $f(x, y)$ with respect to the double Hilbert transform, to be

$$\lim_{\varepsilon,\eta\to 0} \frac{1}{\pi^2}\int_{\varepsilon}^{\pi}\int_{\eta}^{\pi} \frac{[f(x+u,y+v) - f(x-u,y+v) - f(x+u,y-v) + f(x-u,y-v)]}{4\tan(u/2)\tan(v/2)}\,du\,dv.$$

In [3, p. 170] K. Sokół-Sokołowski proved that if $f(x, y)$ is 2π periodic in each variable and belongs to the class L^p, $p > 1$, then $\tilde{f}(x, y)$ exists almost everywhere.

In this paper we shall show that if $f(x, y)$ is sufficiently continuous then

$$\tilde{\sigma}_n(x,y,f) - \int_{1/n}^{\pi}\int_{1/n}^{\pi} \frac{[f(x+u,y+v) - f(x-u,y+v) - f(x+u,y-v) + f(x-u,y-v)]}{4\tan(u/2)\tan(v/2)}\,du\,dv \to 0 \quad \text{uniformly in } (x,y),$$

where $\tilde{\sigma}_n(x, y, f)$ is the first arithmetic mean of the conjugate series.

Before we proceed we shall need the following definitions and inequalities. Define

$$\tilde{f}_{1/n}(x,y) = \frac{1}{\pi^2}\int_{1/n}^{\pi}\int_{1/n}^{\pi} \frac{[f(x+u,y+v) - f(x-u,y+v) - f(x+u,y-v) + f(x-u,y-v)]}{4\tan(u/2)\tan(v/2)}\,du\,dv$$

and

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$$\tilde{\sigma}_n(x,y,f) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u, y+v)\tilde{\sigma}_n(u)\tilde{\sigma}_n(v)\,du\,dv$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}[f(x+u,y+v) - f(x-u,y+v)$$

$$-f(x+u,y-v) + f(x-u,y-v)]\tilde{\sigma}_n(u)\tilde{\sigma}_n(v)\,du\,dv,$$

where

$$\tilde{\sigma}_n(u) = \sum_{k=1}^{n} (\sin(ku))(n+1-k)/(n+1)$$

$$= \tfrac{1}{2}\tan(u/2) - \sin(n+1)u/(n+1)(2\sin(u/2))^2$$

$$= \tfrac{1}{2}\tan(u/2) - H_n(u)$$

is the first arithmetic means of the conjugate Dirichlet kernel. It is well known that

$$\tilde{\sigma}_n(x) \geq 0, \qquad 0 \leq x \leq \pi,$$

$$|\tilde{\sigma}_n(x)| \leq n/2, \quad |\tilde{\sigma}_n(x)| \leq A/x, \quad |H_n(x)| \leq A/(n+1)x^2.$$

THEOREM 1. *Let $f(x,y)$ be a continuous 2π periodic function with modulus of continuity $w_f(\delta) = o(1/\log(1/\delta))$. Then*

$$\lim_{n\to\infty} \tilde{\sigma}_n(a,b,f) - \tilde{f}_{1/n}(a,b) = 0 \quad \textit{uniformly in } (a,b).$$

Without loss of generality let $a = 0$ and $b = 0$. By definition,

$$\tilde{f}_{1/n}(0,0) = \frac{1}{\pi^2}\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{[f(x,y) - f(-x,y) - f(x,-y) + f(-x,-y)]}{4\tan(x/2)\tan(y/2)}\,dx\,dy$$

and

$$\tilde{\sigma}_n(0,0,f) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}[f(x,y) - f(-x,y) - f(x,-y) + f(-x,-y)]$$

$$\cdot\tilde{\sigma}_n(x)\tilde{\sigma}_n(y)\,dx\,dy.$$

Let

$$g(x,y) = f(x,y) - f(-x,y) - f(x,-y) + f(-x,-y)$$

and observe that $g(x,y)$ has a modulus of continuity $w_g(\delta) \leq 4w_f(\delta) = o(1/\log(1/\delta))$. Since $g(x,0) = 0$ for all x and $g(0,y) = 0$ for all y, we have

$$|g(x,y)| \leq \min(o(1/\log(1/x)), o(1/\log(1/y))).$$

We can now write

$$\tilde{\sigma}_n(0,0,f) - \tilde{f}_{1/n}(0,0) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} g(x,y)\tilde{\sigma}_n(x)\tilde{\sigma}_n(y)\,dx\,dy$$

$$(1) \qquad - \frac{1}{\pi^2}\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{g(x,y)}{4\tan(x/2)\tan(y/2)}\,dx\,dy.$$

Since

$$\int_{1/n}^{\pi}\int_{0}^{1/n}|g(x,t)|\tilde{\sigma}_n(x)\tilde{\sigma}_n(y)\,dx\,dy$$
$$\leqslant \int_{1/n}^{\pi}\int_{0}^{1/n}o(1/\log n)(n/2)\tilde{\sigma}_n(y)\,dx\,dy$$
$$\leqslant o(1/\log n)\int_{1/n}^{\pi}\tilde{\sigma}_n(y)\,dy = o(1)$$

and

$$\int_{0}^{1/n}\int_{0}^{1/n}|g(x,y)|\tilde{\sigma}_n(x)\tilde{\sigma}_n(y)\,dx\,dy$$
$$\leqslant o(1/\log n)\int_{0}^{1/n}\int_{0}^{1/n}\frac{n^2}{4}\,dx\,dy$$
$$= o(1/\log n),$$

we can rewrite (1) as follows:

$$o(1) + \frac{1}{\pi^2}\int_{1/n}^{\pi}\int_{1/n}^{\pi}g(x,y)[-(H_n(y)/2\tan(x/2))$$
(2)
$$-(H_n(x)/2\tan(y/2)) + H_n(x)H_n(y)]\,dx\,dy.$$

Choose $0 < \alpha < 1$ and break the integral in (2) into four parts:

(3) $$\int_{1/n^\alpha}^{\pi}\int_{1/n^\alpha}^{\pi} + \int_{1/n^\alpha}^{\pi}\int_{1/n}^{1/n^\alpha} + \int_{1/n}^{1/n^\alpha}\int_{1/n^\alpha}^{\pi} + \int_{1/n}^{1/n^\alpha}\int_{1/n}^{1/n^\alpha}.$$

The first of these is majorized by

(4) $$\int_{1/n^\alpha}^{\pi}\int_{1/n^\alpha}^{\pi}|g(x,y)|\left\{\frac{A}{(n+1)y^2x} + \frac{A}{(n+1)yx^2} + \frac{A}{(n+1)^2x^2y^2}\right\}dx\,dy.$$

Setting $G = \max|g(x,y)|$ we can bound (4) with

$$GA\Big[O(n^\alpha \log n^\alpha/(n+1)) + O(n^\alpha \log n^\alpha/(n+1))$$
$$+O\big(n^{2\alpha}/(n+1)^2\big)\Big] = o(1).$$

The second and third regions in (3) are majorized by

$$\int_{1/n^\alpha}^{\pi}\int_{1/n}^{1/n^\alpha}w_g\left(\frac{1}{n^\alpha}\right)\left[\frac{A}{(n+1)y^2x} + \frac{A}{(n+1)yx^2} + \frac{A}{(n+1)^2x^2y^2}\right]dx\,dy$$
$$= w_g\left(\frac{1}{n^\alpha}\right)\Big[O(n^\alpha \log n/(n+1)) + O(n\log n^\alpha/(n+1))$$
$$+O\big(n^\alpha n/(n+1)^2\big)\Big]$$
$$= o(1).$$

The fourth region is majorized by

$$\int_{1/n}^{1/n^\alpha}\int_{1/n}^{1/n^\alpha} w_g\left(\frac{1}{n^\alpha}\right)\left[\frac{A}{(n+1)y^2x}+\frac{A}{(n+1)xy^2}+\frac{A}{(n+1)^2x^2y^2}\right]dx\,dy$$

$$= w_g\left(\frac{1}{n^\alpha}\right)\Big[O(n\log n/(n+1)) + O(n\log n/(n+1)) + O\big(n^2/(n+1)^2\big)\Big]$$

$$= o(1).$$

Therefore

$$\lim_{n\to\infty} \tilde{\sigma}_n(0,0,f) - \tilde{f}_{1/n}(0,0) = 0.$$

In order to show this theorem is best possible, we shall construct a continuous 2π periodic function f with modulus of continuity $w_f(\delta) = O(1/\log(1/\delta))$ for which $\lim_{n\to\infty}\tilde{\sigma}_n(0,0,f) - \tilde{f}_{1/n}(0,0) \neq 0$.

Let

$$n_k = 2^{(2^k)} + 1,$$

$$g_k(x) = \begin{cases} 1/\log(1/(x-\pi/n_k)), & \pi/n_k \leqslant x \leqslant 3\pi/2n_k, \\ 1/\log(1/(2\pi/n_k - x)), & 3\pi/2n_k < x \leqslant 2\pi/n_k, \\ 0 & \text{otherwise,} \end{cases}$$

and $h(x) = \sum_{k=1}^{\infty} g_k(x)$. Since the support of the $g_k(x)$, $k = 1, \ldots,$ are disjoint, the modulus of continuity $w_h(\delta)$ of $h(x)$ is equal to $\sup_k w_{g_k}(\delta)$. But each $g_k(x)$ has a modulus of continuity $w_{g_k}(\delta) = 1/\log(1/\delta)$; therefore $w_h(\delta) = 1/\log(1/\delta)$. Next we define

$$f(x,y) = \begin{cases} h(y) & 0 \leqslant x \leqslant \pi, 0 \leqslant y \leqslant x, \\ h(x), & 0 \leqslant x \leqslant \pi, x < y \leqslant \pi, \\ 0, & -\pi \leqslant x < 0, \text{ or } -\pi \leqslant y < 0, \\ 2\pi \text{ periodic}, & \text{otherwise.} \end{cases}$$

The function $f(x, y)$ is a continuous 2π periodic function with modulus of continuity $w_f(\delta) = 1/\log(1/\delta)$.

Since $f(x, y)$ is 0 for $-\pi \leqslant x < 0$ or $-\pi \leqslant y < 0$,

$$\tilde{\sigma}_n(0,0,f) - \tilde{f}_{1/n}(0,0) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\tilde{\sigma}_n(x)\tilde{\sigma}_n(y)\,dx\,dy - \frac{1}{\pi^2}\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{f(x,y)}{4\tan(x/2)\tan(y/2)}\,dy\,dx.$$

Since $f(x, y)\tilde{\sigma}_n(x)\tilde{\sigma}_n(y) \geqslant 0$ for $\{(x, y)|0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi\}$,

$$\tilde{\sigma}_n(0,0,f) - \tilde{f}_{1/n}(0,0)$$

$$\geqslant \frac{1}{\pi^2}\int_{1/n}^{\pi}\int_{1/n}^{\pi} f(x,y)\left[\tilde{\sigma}_n(x)\tilde{\sigma}_n(y) - \tfrac{1}{4}\cot(x/2)\cot(y/2)\right]dy\,dx.$$

For $\frac{1}{2} < \alpha < 1$ we shall break the integration into four regions.

From the proof of Theorem 1 we have that the first of these regions is $O(n^{2\alpha}/n^2)$. Because $f(x,y)$ is symmetric about the line $y = x$, the integrals of the second and third regions are equal. In the second region

$$\int_{1/n^\alpha}^{\pi}\int_{1/n}^{1/n^\alpha} f(x,y)\left[\tilde{\sigma}_n(x)\tilde{\sigma}_n(y) - \tfrac{1}{4}\cot(x/2)\cot(y/2)\right] dy\, dx$$

$$= \int_{1/n^\alpha}^{\pi}\int_{1/n}^{1/n^\alpha} h(y)\left[\frac{-H_n(x)}{2\tan(y/2)} - \frac{H_n(y)}{2\tan(x/2)} + H_n(x)H_n(y)\right] dy\, dx. \tag{5}$$

Breaking the right-hand side of (5) into three parts we have that the first of these is

$$-\int_{1/n^\alpha}^{\pi}\int_{1/n}^{1/n^\alpha} \frac{h(y)H_n(x)}{2\tan(y/2)}\, dy\, dx$$
$$= \int_{1/n^\alpha}^{\pi} -H_n(x)\, dx \int_{1/n}^{1/n^\alpha} \frac{h(y)}{2\tan(y/2)}\, dy. \tag{6}$$

From its definition $h(0)=0$ and $h(y) \leqslant 1/\log(1/y)$. Therefore (6) is majorized by

$$O(n^\alpha/n)O\int_{1/n}^{1/n^\alpha} 1/y\log(1/y)\, dy$$
$$= O(n^\alpha/n)O(-\log\log n^\alpha + \log\log n)$$
$$= O(1/n^{1-\alpha}).$$

The third part of (5) is majorized by

$$w_h(\pi)\int_{1/n^\alpha}^{\pi}\int_{1/n}^{1/n^\alpha} \frac{A^2}{(n+1)^2x^2y^2}\, dy\, dx = O(1/n^{1-\alpha}).$$

The second part of (5) is

$$-\int_{1/n^\alpha}^{\pi}\int_{1/n}^{1/n^\alpha} \frac{h(y)H_n(y)}{2\tan(x/2)}\, dy\, dx$$
$$= -\alpha(O(1) + \log n)\int_{1/n}^{1/n^\alpha} h(y)H_n(y)\, dy. \tag{7}$$

For $n = n_k - 1$ where $k > k_0(\alpha)$ sufficiently large,

$$h(y) = \begin{cases} 0, & 1/n \leqslant y \leqslant \pi/n_k, \\ g_k(y), & \pi/n_k \leqslant y \leqslant 2\pi/n_k, \\ 0, & 2\pi/n_k < y \leqslant 1/n^\alpha. \end{cases} \tag{8}$$

Therefore (7) becomes

$$
\begin{aligned}
&\frac{-\alpha(O(1)+\log n)}{n+1}\int_{\pi/n_k}^{2\pi/n_k}\frac{g_k(y)\sin(n+1)y}{(2\sin(y/2))^2}\,dy\\
&\qquad \geqslant \frac{\alpha(O(1)+\log n)}{n+1}\int_{5\pi/4n_k}^{7\pi/4n_k}\frac{g_k(5\pi/4n_k)(-\sin(5\pi/4))}{(2\sin(y/2))^2}\,dy\\
&\qquad \geqslant \frac{\alpha(O(1)+\log n)(-\sin(5\pi/4))}{(n+1)\log(4n_k/\pi)}\int_{5\pi/4n_k}^{7\pi/4n_k}\frac{A}{y^2}\,dy\\
&\qquad \geqslant c>0.
\end{aligned}
$$

From symmetry and (8) we have that the 4th part is

$$
\begin{aligned}
&\int_{\pi/n_k}^{2\pi/n_k}\int_{y}^{2\pi/n_k} \qquad (9)\\
&\qquad + \int_{\pi/n_k}^{2\pi/n_k}\int_{2\pi/n_k}^{1/n^\alpha} 2h(y)\left[\frac{-H_n(y)}{2\tan(x/2)}-\frac{H_n(x)}{2\tan(y/2)}+H_n(x)H_n(y)\right]dx\,dy.
\end{aligned}
$$

Since $H_n(x) \leqslant 0$ for $\pi/n_k \leqslant x \leqslant 2\pi/n_k$, both the first integral in (9) and the first term in the second integral are positive. Therefore (9) is greater than

$$\int_{\pi/n_k}^{2\pi/n_k}\int_{2\pi/n_k}^{1/n^\alpha} 2h(y)\left[\frac{-H_n(x)}{2\tan(y/2)}+H_n(x)H_n(y)\right]dx\,dy$$

whose modulus is majorized by

$$O(-\log\log(2\pi/n_k)+\log\log(\pi/n_k))+O(1/\log n)=o(1).$$

Combining the above results gives

$$\tilde{\sigma}_n(0,0,f)-\tilde{f}_{1/n}(0,0)\geqslant o(1)+c \quad \text{for } n=2^{(2^k)}-1.$$

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