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SOME REMARKS ON SUMMABILITY FACTORS[1]

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ABSTRACT. Bosanquet [2] showed that a necessary and sufficient condition for $\sum_{k=1}^{\infty} x_k y_k$ to be Cesàro summable of order n (n is a nonnegative integer) whenever $\sigma_k^n(y) = O(k)$ where $\sigma_k^n(y)$ is the k th Cesàro mean of y of order n is that $\sum_{k=1}^{\infty} k^{n+1}|\Delta^{n+1}x_k| < \infty$ and $\lim_{k\to 0} kx_k = 0$. The main result of this paper is to show that a necessary and sufficient condition for $\sum_{k=1}^{\infty} x_k y_k$ to be Cesàro summable of order n (n is a nonnegative integer) whenever $\sum_{k=1}^{\infty} k^{n+1}|\Delta^{n+1}x_k| < \infty$ and $\lim_{k\to\infty} kx_k = 0$ is that $\sigma_k^n(y) = O(k)$.

Introduction. Let the linear space of all complex sequences be denoted by ω. Any linear subspace of ω is called a sequence space. A sequence space which is a Banach space such that for $k = 1, 2, \ldots$ the linear functionals f_k, where $f_k(x) = x_k$ are continuous, is called a BK-space. Let m be a nonnegative integer, and E a Banach space containing $e_n = \{\delta_{nk}\}_{k=1}^{\infty}$, $n = 1, 2, \ldots$. If each $x \in E$ has the property that

$$\sigma_n^m(x) = \sum_{k=1}^{n} \frac{1}{\binom{m+n}{n}} \binom{m+n-k}{m} x_k e_k \in E \quad \text{for} \quad n = 1, 2, \ldots$$

and $x = \lim_{n\to\infty} \sigma_n^m(x)$, then E is called a BK-space with $(C,m) - AK$. It is well known that $(C,m) - AK$ implies $(C,m+k) - AK$ for $k = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$.

Let E and F be sequence spaces. $(E \to F) = \{x \in \omega \colon xy = \{x_k y_k\}_{k=1}^{\infty} \in F$ for all $y \in E\}$ is called the space of multipliers from E into F. Associated with each sequence space E are the integrated and differentiated spaces of E defined respectively by

$$\int E = \{x \in \omega \colon dx = \{kx_k\}_{k=1}^{\infty} \in E\},$$

$$dE = \left\{x \in \omega \colon \int x = \left\{\frac{x_k}{k}\right\}_{k=1}^{\infty} \in E\right\}.$$

The following sequence spaces will be considered.

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$l^\infty = \{x \in \omega\colon \sup_k |x_k| < \infty\}$,
$l = \{x \in \omega\colon \sum_{k=1}^\infty |x_k| < \infty\}$,
$c_0 = \{x \in \omega\colon \lim_{k\to\infty} x_k = 0\}$,
$bv_0 = \{x \in \omega\colon \sum_{k=1}^\infty |x_k - x_{k+1}| < \infty \text{ and } \lim_{k\to\infty} x_k = 0\}$,
$q_0 = \{x \in \omega\colon \sum_{k=1}^\infty (k+1)|\Delta^2 x_k| < \infty \text{ and } \lim_{k\to\infty} x_k = 0\}$,
$\sigma_\infty = \{x \in \omega\colon \sup_k |k^{-1}\sum_{n=1}^k x_n| < \infty\}$.

For $x \in \omega$, set

$$s_n^0(x) = x_1 + x_2 + \cdots + x_n,$$

$$s_n^1(x) = s_1^0(x) + s_2^0(x) + \cdots + s_n^0(x),$$

$$\vdots$$

$$s_n^k(x) = s_1^{k-1}(x) + s_2^{k-1}(x) + \cdots + s_n^{k-1}(x);\ k, n = 1, 2, 3, \ldots.$$

Set $\sigma_n^k(x) = s_n^k(x)/\binom{n+k}{k}$, $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$. $\sigma_n^k(x)$ is called the nth Cesàro mean of x of order k. If $\lim_{n\to\infty} \sigma_n^k(x) = s$, we say $\sum_{n=1}^\infty x_n$ is (C, k) summable and denote this by $(C,k) - \sum_{n=1}^\infty x_n = s$.

For $k = 0, 1, 2, \cdots$ define
$C_k = \{x \in \omega\colon \lim_{n\to\infty} \sigma_n^k(x) \text{ exists}\}$,
$'C_k = \{x \in \omega\colon \sup_n |\sigma_n^k(x)| < \infty\}$, and
$\sigma_{k,\infty} = \{x \in \omega\colon \sup_n |n^{-1}\sigma_n^k(x)| < \infty\}$.
$h_b^{k+1} = \{x \in \omega\colon \sum_{n=1}^\infty n^{k+1}|\Delta^{k+1} x_n| < \infty\}$ and $h^{k+1} = h_b^{k+1} \cap \int c_0$ where
$\Delta^1 x_n = x_n - x_{n+1}$ and $\Delta^{k+1} x_n = \Delta^1(\Delta^k x_n)$, $k = 1, 2, 3, \ldots$,
$v^{k+1} = \{x \in \omega\colon \sum_{n=1}^\infty (n+1)^k |\Delta^{k+1} x_n| < \infty\}$, and
$v_0^{k+1} = v^{k+1} \cap c_0$.

Let E be a sequence space and n a nonnegative integer. The set of all $y \in \omega$ such that $(C,n) - \sum_{k=1}^\infty x_k y_k$ exists for all $x \in E$ is called the set of summability factors of E of order n. Clearly the set of summability factors of E of order n is the set of multipliers $(E \to C_n)$. Using the multiplier notation, Bosanquet [2, p. 296, Theorem A] showed that $(\sigma_{n,\infty} \to C_n) = h^{n+1}$ for $n = 0, 1, 2, \ldots$. Our main result stated in multiplier notation is $(h^{n+1} \to C_n) = \sigma_{n,\infty}$ for $n = 0, 1, 2, \ldots$. (See Theorem 2.12.)

1. **Preliminary statements.**

1.1 PROPOSITION. *For* $m = 1, 2, \ldots,\ h^{m+1} \subset h^m$.

PROOF. Let m be a positive integer and let $x \in h^{m+1}$. Then $\sum_{k=1}^\infty k^{m+1}|\Delta^{m+1} x_k| < \infty$ and $\lim_{k\to\infty} x_k = 0$. Set $\varepsilon_k = \Delta^m x_k$; then $\Delta\varepsilon_k = \Delta^{m+1} x_k$. Hence $\sum_{k=1}^\infty k^{m+1}|\Delta\varepsilon_k| = \sum_{k=1}^\infty k^{m+1}|\Delta^{m+1} x_k| < \infty$. By [1, p. 42, Lemma 6] there exists a number s such that
(i) $\varepsilon_k = s + o(1/k^{m+1})$ and
(ii) $\sum_{k=1}^\infty k^m|\varepsilon_k - s| < \infty$.
Since $h^m \subset c_0$,

$$\lim_{k\to\infty} \varepsilon_k = \lim_{k\to\infty} \Delta^m x_k = 0. \tag{1.2}$$

By (1.2) and (i), $s = 0$. By (ii) $\sum_{k=1}^\infty k^m|\Delta^m x_k| < \infty$. Hence $x \in h^m$.

REMARK. By [5, p. 96, Theorem 3.2], $h^1 \subset l$. Hence $h^m \subset l$ for $m = 1, 2, 3, \ldots$. (See 1.1.)

The following results will be used:

1.3. If E and F are BK-spaces with $(C,k) - AK$, then $(E \cap F \to C_k) = (E \to C_k) + (F \to C_k)$ [4, p. 156, Theorem 4].

1.4. If $r > -1$, $p \geqslant 0$, $\sum_{k=1}^{\infty} k^{p-1}|x_k| < \infty$, then $\sum_{k=1}^{\infty} k^{p+r}|\Delta^{r+1}x_k| < \infty$ if and only if $\sum_{k=1}^{\infty} k^{p+r+1}|\Delta^{r+1}(x_k/k)| < \infty$ [3, p. 77].

If $p = 0$ in 1.4, we get for $r = 1, 2, \ldots,$ $dl \cap dh_b^{r+1} = dl \cap v^{r+1}$. Hence for $r = 1, 2, \ldots,$ $l \cap h_b^{r+1} = l \cap \int v^{r+1}$. Since $h^{r+1} \subset l$ for $r = 0, 1, 2, \ldots,$

$$h^{r+1} = \int v_0^{r+1} \cap l. \tag{1.5}$$

2. **The space of multipliers from h^n into C_{n-1}.**

2.1 LEMMA. $l^\infty \subset \sigma_{n-1,\infty}$, $n = 1, 2, \ldots$.

PROOF. The proof is by induction. Since $\sigma_{0,\infty} = \sigma_\infty$ the statement is true for $n = 1$. Assume

$$l^\infty \subset \sigma_{k,\infty} \quad \text{for } 1 \leqslant k < n, \tag{2.2}$$

$$\left|\frac{s_r^n(x)}{r^{n+1}}\right| < \frac{1}{r}\sum_{j=1}^{r}\left|\frac{s_j^{n-1}(x)}{j^n}\right|. \tag{2.3}$$

If $x \in l^\infty$, then by (2.2) $\sup_j |s_j^{n-1}(x)/j^n| \leqslant M < \infty$. Hence (2.3) implies $\sup_r |s_r^n(x)/r^{n+1}| \leqslant M$. Thus $l^\infty \subset \sigma_{n,\infty}$.

2.4 LEMMA. *For $k = 0, 1, 2, \ldots, n = 1, 2, \ldots$ and $x \in \omega$,*

$$s_n^k(dx) = n s_n^k(x) - (k+1)s_{n-1}^{k+1}(x).$$

PROOF. Follows immediately from Abel's partial summation and definition of $s_n^k(x)$.

2.5 PROPOSITION. *For $k = 0, 1, 2, \ldots,$ $d'C_k \subset \sigma_{k,\infty}$.*

PROOF. It suffices to show that $'C_k \subset \int \sigma_{k,\infty}$ for $k = 0, 1, 2, \ldots$. Let $x \in' C_k$, then

$$\sup_n \left|\frac{s^k(x)}{n^k}\right| = M < \infty. \tag{2.6}$$

Since $'C_k \subset {}'C_{k+1}$ for $k = 0, 1, 2, \ldots,$

$$\sup_n \left|\frac{s_n^{k+1}(x)}{n^{k+1}}\right| = M' < \infty \tag{2.7}$$

for some $M' > 0$. From 2.4 we obtain

$$s_n^k(dx) = n s_n^k(x) - (k+1)s_{n-1}^{k+1}(x) \quad \text{for } n = 1, 2, \ldots. \tag{2.8}$$

Now (2.6), (2.7) and (2.8) imply

$$\left|\frac{s_n^k(dx)}{n^{k+1}}\right| \leqslant \left|\frac{s_n^k(x)}{n^k}\right| + (k+1)\left|\frac{s_{n-1}^{k+1}(x)}{n^{k+1}}\right| \leqslant M + (k+1)M'$$

for all n. Hence $dx \in \sigma_{k,\infty}$, i.e., $x \in \int \sigma_{k,\infty}$.

2.9 THEOREM. *For* $n = 0, 1, 2, \ldots,\quad \sigma_{n,\infty} = ((\sigma_{n,\infty} \to C_n) \to C_n)$.

PROOF. Evidently $\sigma_{n,\infty} \subseteq ((\sigma_{n,\infty} \to C_n) \to C_n)$ for all n. Bosanquet proved that $(\sigma_{n,\infty} \to C_n) = h^{n+1}$ for $n = 0, 1, 2, \ldots$ [2, p. 296, Theorem A]. For $n = 0, 1, 2, \ldots, h^{n+1} = \int v_0^{n+1} \cap l$, $n = 0, 1, 2, \ldots$. By [4, p. 155, Theorem 3], $\int v_0^{n+1}$ is a BK-space with $(C, n) - AK$. Since l is a BK-space with AK, it also has $(C, n) - AK$. From [4, p. 165, Theorem 4]

$$\begin{aligned}((\sigma_{n,\infty} \to C_n) \to C_n) &= \left(\int v_0^{n+1} \cap l \to C_n\right) \\ &= \left(\int v_0^{n+1} \to C_n\right) + (l \to C_n) \quad \text{for} \quad n = 0, 1, 2, \ldots.\end{aligned}$$

$(\int v_0^{n+1} \to C_n) = d'C_n, n = 0, 1, 2, \ldots$ [4, p. 156, Theorem 5]. Since l has AK, $(l \to C_n) = (l \to C_0) = l^\infty$. Thus, by 2.1 and 2.5,

$$((\sigma_{n,\infty} \to C_n) \to C_n) = d'C_n + l^\infty \subset \sigma_{n,\infty}. \tag{2.10}$$

Hence $((\sigma_{n,\infty} \to C_n) \to C_n) = \sigma_{n,\infty}$ for $n = 0, 1, 2, \ldots$.

2.11 COROLLARY. *For* $n = 0, 1, 2, \ldots,\quad \sigma_{n,\infty} = d'C_n + l^\infty$.

PROOF. See (2.10) in proof of 2.9.

Our main result can now be stated and proved.

2.12 THEOREM. *For* $n = 1, 2, \ldots, (h^n \to C_{n-1}) = \sigma_{n-1,\infty}$.

PROOF. By 2.9 $\sigma_{n,\infty} = ((\sigma_{n,\infty} \to C_n) \to C_n)$ for $n = 0, 1, 2, \ldots,$

$$h^n = (\sigma_{n-1,\infty} \to C_{n-1}) \quad \text{for } n = 1, 2, \ldots$$

[2, p. 296, Theorem A]. Hence

$$(h^n \to C_{n-1}) = ((\sigma_{n-1,\infty} \to C_{n-1}) \to C_{n-1}) \quad \text{for } n = 1, 2, \ldots.$$

2.13 COROLLARY. *For* $n = 1, 2, \ldots, ((h^n \to C_{n-1}) \to C_{n-1}) = h^n$.

PROOF. Let n be a positive integer. $(h^n \to C_{n-1}) = \sigma_{n-1,\infty}$ by 2.12.

$$(\sigma_{n-1,\infty} \to C_{n-1}) = h^n$$

by [2, p. 296, Theorem A].

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