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WHICH OPERATORS ARE SIMILAR TO PARTIAL ISOMETRIES?

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ABSTRACT. Let $\mathcal{H}$ denote a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{P} = \{T \text{ in } \mathcal{L}(\mathcal{H}) | r(T) < 1 \text{ and } T \text{ is similar to a partial isometry with infinite rank}\}$; let $\mathcal{S} = \{S \text{ in } \mathcal{L}(\mathcal{H}) | r(S) < 1, \text{range}(S) \text{ is closed, and } \text{rank}(S) = \text{nullity}(S) = \text{corank}(S) = \aleph_0\}$. It is conjectured that $\mathcal{P} = \mathcal{S}$ and it is proved that $\mathcal{P} \subset \mathcal{S} \subset \mathcal{P}^-$.

Introduction. Let $\mathcal{H}$ denote a fixed separable, infinite-dimensional complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. In [5], Sz.-Nagy proved that an invertible operator T in $\mathcal{L}(\mathcal{H})$ is similar to a unitary operator if and only if the powers of T and T^{-1} are uniformly bounded; the proof of this result also implies that an operator is similar to an isometry if and only if its powers are uniformly bounded above and below [4]. In this note we state the following conjecture concerning operators similar to partial isometries, and then prove results which partially affirm the conjecture.

CONJECTURE. If T is an operator on $\mathcal{H}$ with closed range, whose spectral radius is less than one, and such that $\text{rank}(T) = \text{nullity}(T) = \text{nullity}(T^*) = \aleph_0$, then T is similar to a partial isometry.

Let $\mathcal{P} = \{T \text{ in } \mathcal{L}(\mathcal{H}) | r(T) < 1 \text{ and } T \text{ is similar to a partial isometry with infinite rank}\}$, where $r(T)$ is the spectral radius of T; let $\mathcal{S} = \{S \text{ in } \mathcal{L}(\mathcal{H}) | r(S) < 1, \text{range}(S) \text{ is closed, and } \text{rank}(S) = \text{nullity}(S) = \text{corank}(S) = \aleph_0\}$. It is easy to prove that $\mathcal{P} \subset \mathcal{S}$ and in this note we prove that $\mathcal{P} \subset \mathcal{S} \subset \mathcal{P}^-$ (the norm closure of $\mathcal{P}$ in $\mathcal{L}(\mathcal{H})$). To state the results in detail we use the following notation. If A and B are operators on $\mathcal{H}$ such that $A^*A + B^*B$ is invertible, let $M(A,B)$ denote the operator on $\mathcal{H} \oplus \mathcal{H}$ whose matrix is $\left(\begin{smallmatrix} 0 & A \\ 0 & B \end{smallmatrix}\right)$; let $\mathcal{T}$ denote the set of all matrices of this form whose spectral radii are less than one. Each operator in $\mathcal{S}$ is unitarily equivalent to a matrix in $\mathcal{T}$.

THEOREM 1. *The operator $M(A, B)$ in $\mathcal{T}$ is similar to a partial isometry if any of the following conditions are satisfied*:

(i) 0 *is not in the interior of* $\sigma(A)$;

(ii) $\text{nullity}(A) = \text{corank}(A)$;

(iii) $\text{nullity}(A) < \text{corank}(A) = \aleph_0$ *and B is not compact*;

(iv) *B is a semi-Fredholm operator*;

(v) $\text{corank}(A) < \text{nullity}(A)$, *$A$ has closed range, and $B^*|E$ is not compact, where*

$$E = \{y \in \mathcal{H} | \exists x \in \mathcal{H} \ni: A^*x + B^*y = 0\}.$$

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Let (J) denote the ideal of all compact operators in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. If T is in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, let $\tilde{T}$ denote the image of T under the canonical homomorphism of $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})/(J)$.

THEOREM 2. *If $T = M(A,B)$ is in $\mathfrak{T}$ then $\tilde{T}$ is similar to a partial isometry if either of the following conditions is satisfied:*

(i) nullity(A) *and* corank(A) *are finite*;

(ii) *B is compact.*

Because these results do not cover the case corank(A) $<$nullity$(A) = \aleph_0$, we do not know whether $\mathcal{P} = \mathcal{S}$. We note also that the proof of Theorem 1-iii was motivated by the proof of a factorization theorem of R. G. Douglas [1, Lemma 2.1]. The author thanks the referee for suggestions that have clarified certain points in the original proofs of our results.

Proof of Theorems 1 and 2.

LEMMA 0. *If $T = M(A,B)$ is in $\mathfrak{T}$, then the nonzero elements of $\sigma(T)$ and $\sigma(B)$ are identical.*

PROOF. If $\lambda \neq 0$ and $B - \lambda$ is invertible, then a calculation shows that $(T-\lambda)^{-1}$ is given by the operator matrix

$$\begin{pmatrix} -1/\lambda & (-1/\lambda)A(B-\lambda)^{-1} \\ 0 & (B-\lambda)^{-1} \end{pmatrix}.$$

If $\lambda \neq 0$ and the inverse of $T - \lambda$ exists, denote this inverse by the operator matrix $\left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$ a calculation shows that $Z = 0$, so that $W = (B-\lambda)^{-1}$.

LEMMA 1. *If T is in $\mathfrak{T}$, then T is similar to an operator $M(A, B)$ such that* $\|B\| < 1$.

PROOF. If $T = M(A(T), B(T))$, Lemma 0 implies that $r(B(T)) < 1$, and Problem 122 of [3] implies that there exists an invertible operator X such that $\|XB(T)X^{-1}\| < 1$. Since T is similar to $M = M(A(T)X^{-1}, XB(T)X^{-1})$, the proof is complete.

LEMMA 2. *If T is in $\mathfrak{T}$ and* nullity$(A(T))=$ corank$(A(T))$, *then T is similar to an operator $M(A, B)$ such that $A \geqq 0$ and $\|B\| < 1$.*

PROOF. Consider the operator M of Lemma 1. We have nullity$(A(T)X^{-1})$ $=$nullity$(A(T)) =$ corank$(A(T)) =$ corank$(A(T)X^{-1})$, and thus $A(T)X^{-1} = UP$, where U is unitary and $P \geqq 0$. Since M is unitarily equivalent to $M(P, XB(T)X^{-1})$, the proof is complete.

LEMMA 3. *Let $T = M(A,B)$ be in $\mathfrak{T}$ and suppose $A^*A + B^*B \geqq \varepsilon^2 > 0$. If $|\lambda| > 1$, then T is similar to $M(A - \varepsilon/\lambda, B)$.*

PROOF. Theorem 1 of [1] implies that there exist operators X_1 and X_2 such that $X_1A + X_2B = \varepsilon$ and $X_1^*X_1 + X_2^*X_2 \leqq 1$. Let $|\lambda| > 1$ and let S denote the operator on $\mathcal{H} \oplus \mathcal{H}$ whose matrix is

$$\begin{pmatrix} X_1 - \lambda & X_2 \\ 0 & -\lambda \end{pmatrix}.$$

Now S is invertible and a calculation shows that $SM(A,B)S^{-1}$ is of the desired form.

PROOF OF THEOREM 1-i. The operator M of Lemma 1 is similar to $M(XA(T)X^{-1}, XB(T)X^{-1})$, and thus we may assume that $\|B\| < 1$ and 0 is not in the interior of $\sigma(A)$. By an application of Lemma 3 with λ suitably chosen such that $A - \varepsilon/\lambda$ is invertible and $|\lambda| > 1$, we may assume that A is invertible. Since $\|B\| < 1$, we may define $R = A(1 - B^* B)^{-1/2}$ and $S = R \oplus 1_{\mathcal{H}}$; a calculation shows that $S^{-1}TS = M((1 - B^* B)^{1/2}, B)$, which is a partial isometry, and therefore the proof is complete.

PROOF OF THEOREM 1-ii. We may assume from Lemma 2 that $A \geqq 0$; the result now follows from Theorem 1-i.

PROOF OF THEOREM 1-iii. Recall that an operator B in $\mathcal{L}(\mathcal{H})$ is not compact if and only if the range of B contains a closed, infinite-dimensional subspace (see, for example, Theorem 2.5 of [2] and Problem 141 of [3]). It follows from this fact and an application of the open mapping theorem that B is not compact if and only if B is bounded below on some closed, infinite-dimensional subspace $M \subset \ker(B)^{\perp}$. Thus there exists $\delta > 0$ such that $\|Bm\| \geqq \delta\|m\|$ for all m in M. For each m in M, we set $X_1(Bm) = Am$. Now

$$\|X_1(Bm)\| = \|Am\| \leqq \|A\|\,\|m\| \leqq (\|A\|/\delta)\|Bm\|,$$

and it follows that X_1 is a well-defined bounded linear operator defined on the closed subspace $B(M)$. Let Q denote the projection onto $B(M)$, and let $X = X_1 Q$ in $\mathcal{L}(\mathcal{H})$. Now $M \subset \ker(A - XB)$ and since $(A - XB)\mathcal{H} \subset A\mathcal{H}$, we have $\dim\ker(A - XB) = \dim\ker((A - XB)^*) = \aleph_0$. Since T is similar to $M(A - XB, B)$, the proof may be completed by an application of Theorem 1-ii.

PROOF OF THEOREM 1-iv. From Lemma 1, we may assume $\|B\| < 1$. Recall that an operator B in $\mathcal{L}(\mathcal{H})$ is semi-Fredholm if B has closed range and if either nullity(B) or corank(B) is finite. We consider first the case nullity$(B) < \aleph_0$; there exists an operator L and a finite rank operator K such that $LB = 1 + K$. Let $X = (\sqrt{1 - B^* B} - A)L$ and let S denote the operator on $\mathcal{H} \oplus \mathcal{H}$ whose matrix is $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)$. A calculation shows that $STS^{-1} = M(\sqrt{1 - B^* B} + J, B)$, where J is a finite rank operator. Since $\|B\| < 1$, $\sqrt{1 - B^* B} + J$ is Fredholm with index equal to zero, and the proof may be completed by an application of Theorem 1-ii.

We now consider the case $\operatorname{corank}(B) < \aleph_0$. In this case B^* has finite nullity and closed range. Let P denote the projection onto the initial space of B^* and let $\mathcal{E} = \{x \in \mathcal{H} | \exists y \in P\mathcal{H} \text{ such that } A^* x + B^* y = 0\}$. Since B^* has closed range, $\mathcal{E}$ is closed; since nullity $(T^*) = \aleph_0$ and nullity $(B^*) < \aleph_0$. It follows readily that $\mathcal{E}$ is infinite dimensional. For each x in $\mathcal{E}$ there is a unique vector $X_1(x)$ in $P\mathcal{H}$ such that $A^* x + B^* X_1(x) = 0$. Since B^* is bounded below on $P\mathcal{H}$, the assignment $x \to X_1(x)$ is bounded and linear on the closed subspace $\mathcal{E}$. Let Q denote the projection onto $\mathcal{E}$ and let $X = X_1 Q$ in $\mathcal{L}(\mathcal{H})$; thus $\mathcal{E} \subset \ker(A + X^* B^*)^*$. Since $\mathcal{E}$ is infinite dimensional and B is not compact, the proof may be completed by an application of Theorem 1-ii-iii.

COROLLARY. $\mathcal{T} \subset \mathcal{P}^{-}$.

PROOF. The preceding result implies that if T is in $\mathcal{T}$ and $B(T)$ is either left or right invertible, then T is in $\mathcal{P}$. Now there exists a sequence $\{B_k\} \subset \mathcal{L}(\mathcal{H})$ such that $\lim \|B_k - B(T)\| = 0$ and such that the sequence elements are either all left invertible or all right invertible [3, Problem 109]. Since $B_k^* B_k + A^*A \to B^*B + A^*A$, we may assume that each $B_k^* B_k + A^*A$ is invertible; from the upper semicontinuity of the spectrum we may assume each $r(B_k) < 1$. Therefore, Theorem 1-iv implies that each $M(A, B_k)$ is in $\mathcal{P}$, and the proof is complete.

We now assume that T is in $\mathcal{T}$ and that A^* has closed range and finite nullity. Let E be as in Theorem 1-v; the hypotheses imply that E is a closed, infinite-dimensional subspace. In view of the previous results it is natural to attempt to find an operator X such that $\operatorname{corank}(A + XB) = \aleph_0$; the following result proves Theorem 1-v.

PROPOSITION. *There exists an operator X such that* $\operatorname{corank}(A + XB) = \aleph_0$ *if and only if $B^* | E$ is not compact.*

PROOF. If $B^*|E$ is not compact, the operator X may be constructed by a straightforward modification of the proof of Theorem 1-iii; details are omitted.

For the converse, we assume that $B^*|E$ is compact. Suppose that there is an operator X on $\mathcal{H}$ and a closed, infinite-dimensional subspace $K \subset \mathcal{H}$ such that $A^*t = B^*X^*t$ for each t in K. Since $\dim \ker(A^*) < \aleph_0$, it follows that $L = K \cap \operatorname{range}(A)$ is infinite dimensional. Since A^* has closed range, A^* is bounded below on L. Let $\{t_n\}$ denote an orthonormal basis for L. Now $t_n \xrightarrow{w} 0$, $\{X^*(t_n)\} \subset E$, and thus $B^*X^*t_n \to 0$. Therefore $A^*t_n \to 0$, which is a contradiction.

PROOF OF THEOREM 2-i. Let $A = UP$ denote the polar decomposition of A. Since $P^2 + B^*B$ is invertible, we may define $T_1 = M(P, B)$, and Lemma 0 implies that $r(T_1) = r(B) = r(M(A, B)) < 1$. Theorem 1-ii now implies that T_1 is similar to a partial isometry. Since the nullity and corank of U are finite, $\tilde{U}$ is unitary, and the proof is completed by noting that

$$T_1 - (U^* \oplus 1)T(U \oplus 1)$$

is of finite rank.

PROOF OF THEOREM 2-ii. Theorem 1 of [1] implies that there exist operators X_1 and X_2 such that $X_1A + X_2B = 1$. Since B is compact, we have $\tilde{X}_1\tilde{A} = 1$, and thus A has closed range and finite nullity. If $A = UP$ denotes the polar decomposition of A, then $P = Q \oplus 0$, where Q is invertible. Set $R = Q^{-1} \oplus 1_{\ker(P)}$ and $S = 1_{\mathcal{H}} \oplus R$. Now $S^{-1}TS$ has the operator matrix

$$\begin{pmatrix} 0 & U \\ 0 & R^{-1}BR \end{pmatrix},$$

which is the sum of a partial isometry and a compact operator.

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