PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 56, April 1976

# INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. This paper is concerned with a class of linear operators acting in the space of the trigonometric polynomials and preserving the inequalities of the form $|S(\theta)| < |T(\theta)|$ in the half plane $\operatorname{Im} \theta > 0$. Some inequalities for entire functions of exponential type and some theorems concerning the distribution of the zeros of the trigonometric polynomials, including an analogue to the Gauss-Lucas theorem, are derived.

1. **Introduction.** Using interpolation series, R. P. Boas [1] obtains the following interesting extension of the classic S. Bernstein inequality:

THEOREM. *Let $f(z)$ be an entire function of exponential type $\sigma$ with $|f(z)| \leqq M$ on the real axis. Then the inequality*

$$(1) \quad |f(x+iy)e^{-iw} + f(x-iy)e^{iw}| \leqq 2M(\cosh^2 \sigma y - \sin^2 w)^{1/2}, \qquad w \text{ real},$$

*holds.*

This theorem, as shown by Boas himself [1], has a number of important consequences. Our purpose is to give a new proof and some extension of (1). At the same time our method, which is based on a principle suggested by a paper of De Bruijn [2], and on a theorem of Obreshkov [3] concerning the zeros of the rational polynomials, allows us to prove some theorems about the zeros of the trigonometric polynomials, including a theorem analogous to the classic Gauss-Lucas theorem.

2. The principle mentioned above is given by

THEOREM 1. *Let $\mathcal{K}$ be a closed subset of the complex plane $C$ and let $\mathfrak{M}$ be a complex linear space of meromorphic functions with poles in $\mathcal{K}$. Further, let $L\colon \mathfrak{M} \to \mathfrak{M}$ be a linear operator and $\mathfrak{N}$ the subset of $\mathfrak{M}$ consisting of the functions having no zeros in $C \backslash \mathcal{K}$. Then the inequality $|f(z)| < |g(z)|$, $z \in C\backslash\mathcal{K}$, $f, g \in \mathfrak{M}$, implies the inequality $|L(f)(z)| < |L(g)(z)|$, $z \in C\backslash\mathcal{K}$, if and only if $L(\mathfrak{N}) \subset \mathfrak{N}$.*

PROOF. Let us suppose that $L(\mathfrak{N}) \subset \mathfrak{N}$ and $|f(z)| < |g(z)|$ in $C\backslash\mathcal{K}$, but nevertheless, there exists $z_0 \in C\backslash\mathcal{K}$ such that $|L(f)(z_0)| \geqq |L(g)(z_0)|$. Intro-

Received by the editors October 4, 1974.
*AMS (MOS) subject classifications* (1970). Primary 30A04, 30A08; Secondary 30A40.
*Key words and phrases.* Inequalities in the complex domain, trigonometric polynomials, extremal problems.

ducing the functions $f-\lambda g$, where $\lambda = L(f)(z_0)[L(g)(z_0)]^{-1}$ ($\lambda$ is well defined because $g \in \mathfrak{N}$ implies $L(g) \neq 0$ in $C\backslash\mathfrak{K}$), we obviously have $f - \lambda g \in \mathfrak{N}$, since $|\lambda| \geqq 1$. Thus $L(f-\lambda g) = L(f) - \lambda L(g) \in \mathfrak{N}$. This is impossible, however, because $L(f-\lambda g)(z_0) = 0$.

Considering the pair of $f$, $g$, where $f \equiv 0$ and $g \in \mathfrak{N}$ is arbitrary, we conclude that the condition $L(\mathfrak{N}) \subset \mathfrak{N}$ is necessary.

In the sequel, except Theorem 1, we shall need a slight modification of the following elementary result due to Obreshkov [3].

THEOREM 2. *Let $\mathfrak{D}$ be the strip bounded by two parallel lines making angles of $\phi$ with the real axis and let all the zeros of the rational polynomial $f(z)$ lie in $\mathfrak{D}$. Then all the zeros of the polynomial*

$$F(z) = f(z+h) - \gamma f(z-h) \quad \textit{where } |\gamma| = 1 \textit{ and } \arg h = \phi + \pi/2,$$

*also lie in $\mathfrak{D}$.*

Because of the importance of this theorem for our considerations, we shall outline the proof of Obreshkov.

PROOF. Let $z_1, z_2, \ldots, z_n$ be the zeros of $f(z)$ and let $z_0$ be a zero of $F(z)$. Then we have

$$\left|\frac{f(z_0+h)}{f(z_0-h)}\right| = 1 \quad \text{i.e.} \quad \prod_{k=1}^{n} \left|\frac{z_0+h-z_k}{z_0-h-z_k}\right| = 1.$$

Supposing for a moment that $z_0$ lies outside $\mathfrak{D}$, we immediately come to a contradiction, because all the factors $|(z_0+h-z_k)/(z_0-h-z_k)|$, $k = 1, 2, \ldots, n$, are simultaneously less than or greater than 1.

The same reasoning proves

THEOREM 3. *Let all the zeros of the rational polynomial $f(z)$ lie in the half plane* $\operatorname{Im} z \leqq a$ *and let $h$ and $k$, $0 \leqq k \leqq h$, $h > 0$, be real numbers. Then the zeros of the polynomial $f(z+hi) - \gamma f(z-ki)$, $|\gamma| \leqq 1$, also lie in* $\operatorname{Im} z \leqq a$.

It will be convenient for our purpose to introduce the following

DEFINITION. A trigonometric polynomial of the form

$$(2) \qquad T(\theta) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu\theta}, \qquad a_{-n} \neq 0,$$

having no zeros in the half plane $H$: $\operatorname{Im}\theta > 0$ is said to belong to class $P_n$.

(In this paper by a trigonometric polynomial of degree not exceeding $n$ we always mean an expression of the form (2) without any restriction on the coefficients.)

REMARK. By means of the substitution $w = e^{i\theta}$ and the maximum principle, it is easily seen that in $H$ the inequality $|\overline{T}(\theta)| \leqq |T(\theta)|$ is satisfied, where $\overline{T}(\theta) = \sum_{\nu=-n}^{n} \bar{a}_\nu e^{-i\nu\theta}$ and $\bar{a}_\nu$ is the conjugate of $a_\nu$. Consequently we have $T(-\theta) \in P$, where $P$ is the class of the majorants studied by B. Levin and others [4, p. 129].

Now we are in a position to prove our main theorem.

THEOREM 4. *Let $S(\theta)$ and $T(\theta)$ be trigonometric polynomials of degree not exceeding $n$ and $T(\theta) \in P_n$. Furthermore, let $S(\theta)$ and $T(\theta)$ be linearly independ-*

*ent and satisfy the inequality* $|S(\theta)| \leqq |T(\theta)|$ *on the real axis. Then the inequality*

$$(3)\quad |S(\theta + \lambda i) - \tau S(\theta - \mu i)| < |T(\theta + \lambda i) - \tau T(\theta - \mu i)|, \qquad \operatorname{Im} \theta > 0,$$

*where* $0 \leqq \mu \leqq \lambda$, $\lambda > 0$ *and* $|\tau| \leqq (\cosh(\lambda/2)/\cosh(\mu/2))^{2n}$, *is satisfied.*

PROOF. First of all by means of the substitution $w = e^{i\theta}$ and the maximum principle, we derive the inequality $|S(\theta)| < |T(\theta)|$ for $\theta \in H$. Furthermore, introducing polynomials $S_1(\theta) = S(\theta + \alpha)$, $T_1(\theta) = T(\theta + \alpha)$, where $\alpha \in H$ is arbitrary and fixed, we obviously have

$$(4)\qquad |S_1(\theta)| < |T_1(\theta)| \quad \text{for } \operatorname{Im} \theta > -\operatorname{Im} \alpha.$$

Now, setting $z = \operatorname{tg}(\theta/2)$, we obtain

$$(5)\qquad S_1(\theta) = P(z)/(1 + z^2)^n, \qquad T_1(\theta) = Q(z)/(1 + z^2)^n,$$

where $P(z)$ and $Q(z)$ are rational polynomials of degree not exceeding $2n$. Since the function $z = \operatorname{tg}(\theta/2)$ maps $H$ to $H \backslash \{i\}$ and $Q(i) = 4^n a_{-n} e^{-in\alpha} \neq 0$, the inequality

$$(6)\qquad |P(z)| < |Q(z)|, \qquad \operatorname{Im} z \geqq 0,$$

follows from (4). Moreover, the relation

$$\lim_{x \to \pm\infty} \left| \frac{P(x)}{Q(x)} \right| = \lim_{\theta \to \pm\pi} \left| \frac{S_1(\theta)}{T_1(\theta)} \right| = \left| \frac{S(\alpha \pm \pi)}{T(\alpha \pm \pi)} \right| < 1, \qquad x \text{ real},$$

implies (6) in the half plane $\operatorname{Im} z \geqq -\varepsilon$, where $\varepsilon > 0$ is sufficiently small.

In order to apply Theorem 1 let us denote by $\mathcal{K}$ the half plane $\operatorname{Im} z \leqq -\varepsilon$, where $\varepsilon > 0$ is chosen so that (6) holds in $C \backslash \mathcal{K}$. Let $\mathfrak{M}$ be the complex space of rational polynomials of degree not exceeding $2n$ and let $\mathfrak{N}$ be the subset of $\mathfrak{M}$ consisting of the polynomials having no zeros outside $\mathcal{K}$. According to Theorem 3, for the operator

$$L(f) = f(z + hi) - \gamma f(z - ki), \qquad 0 \leqq k \leqq h,\ h > 0,\ |\gamma| \leqq 1,\ f \in \mathfrak{M},$$

we have $L(\mathfrak{N}) \subset \mathfrak{N}$. Recalling (6) and applying Theorem 1 we obtain

$$(7)\qquad |P(z + hi) - \gamma P(z - ki)| < |Q(z + hi) - \gamma Q(z - ki)|$$

in $C \backslash \mathcal{K}$ and, in particular, in $\operatorname{Im} z \geqq 0$.

Now let the real numbers $\lambda$, $\mu$, $0 \leqq \mu \leqq \lambda$, $\lambda > 0$, be arbitrary. Setting $z = 0$, $h = \operatorname{tgh}(\lambda/2)$, $k = \operatorname{tgh}(\mu/2)$ in (7), by means of (5) we get

$$\left| S(\alpha + \lambda i) - \gamma \left( \frac{\cosh(\lambda/2)}{\cosh(\mu/2)} \right)^{2n} S(\alpha - \mu i) \right|$$
$$< \left| T(\alpha + \lambda i) - \gamma \left( \frac{\cosh(\lambda/2)}{\cosh(\mu/2)} \right)^{2n} T(\alpha - \mu i) \right|,$$

and since $\alpha \in H$ is arbitrary, the proof of Theorem 4 is complete.

REMARK. If we have $|S(\theta)| < |T(\theta)|$ on the real axis, then (3) is satisfied in the closed half plane $\operatorname{Im}\theta \geqq 0$. Indeed, it suffices to note that the inequality $|S(\theta)| < |T(\theta)|$ holds on the line $\operatorname{Im}\theta = -\varepsilon$, where $\varepsilon > 0$ is sufficiently small, and to apply Theorem 4 to the pair $S(\theta - i\varepsilon)$, $T(\theta - i\varepsilon)$.

COROLLARY 1. *Let us set $\tau = 1$ in* (3) *and, after dividing by $\lambda - \mu$, let $\lambda \to 0$, $\mu \to 0$. Then we get the known inequality* [4]

$$|S'(\theta)| \leqq |T'(\theta)|, \qquad \operatorname{Im}\theta \geqq 0, \tag{8}$$

*which obviously includes Bernstein's inequality. Furthermore, applying Theorem* 4 *with $\tau = e^{2iw}$, $w$ real, to the pair $e^{-iw}S(\theta)$ and $e^{-iw}T(\theta)$, we obtain the inequality*

$$\begin{gathered}|e^{-iw}S(\theta + \lambda i) + e^{iw}S(\theta - \mu i)| < |e^{-iw}T(\theta + \lambda i) + e^{iw}T(\theta - \mu i)|,\\ \operatorname{Im}\theta > 0,\ 0 \leqq \mu \leqq \lambda,\end{gathered} \tag{9}$$

*so that the linear operator $T(\theta) \to e^{-iw}T(\theta + \lambda i) + e^{iw}T(\theta - \mu i)$, acting on the space of the trigonometric polynomials, leaves the class $P_n$ invariant and preserves inequalities of the form $|S(\theta)| < |T(\theta)|$ in the half plane* $\operatorname{Im}\theta > 0$. *Hence it is a B-operator in the sense of Levin* [4, *p.* 226].

COROLLARY 2. *Let us consider the operator*

$$L_f(R)(\theta) = \sum_{\nu=0}^{m} C_\nu R(\theta + (m - \nu)\lambda i - \nu\mu i), \qquad 0 \leqq \mu \leqq \lambda,\ \lambda > 0,$$

*where $f(z) = \sum_{\nu=0}^{m} C_\nu z^{m-\nu}$ is fixed, $R(\theta)$ being a trigonometric polynomial. If all the zeros of $f(z)$ lie in the circle $|z| \leqq (\cosh(\lambda/2)/\cosh(\mu/2))^{2n}$, the inequality*

$$|L_f(S)(\theta)| < |L_f(T)(\theta)|, \qquad \operatorname{Im}\theta > 0, \tag{10}$$

*is satisfied.*

One may prove this corollary by applying Theorem 4 successively with $\tau = \gamma_k$, $k = 1, 2, \ldots, m$, where $\{\gamma_k\}$ are the zeros of $f(z)$.
In turn, inequality (10) implies

COROLLARY 3. *If $T(\theta) \in P_n$, then $L_f(T)(\theta) \in P_n$.*

PROOF. Applying (10) to the pair $S(\theta) \equiv 0$, $T(\theta)$, we see that $L_f(T)(\theta)$ has no zeros in the half plane $\operatorname{Im}\theta > 0$. Since $L_f(T)(\theta)$ obviously has the form (2), the corollary is proved.

Now we may state a theorem analogous to a theorem of L. Weisner [6].

THEOREM 5. *If $T(\theta) \in P_n$, then*

$$L(T)(\theta) = \int_{\theta - \mu i}^{\theta + \lambda i} T(t)\,dt, \qquad 0 \leqq \mu \leqq \lambda,\ \lambda > 0, \tag{11}$$

*also belongs to $P_n$.*

PROOF. First of all it is immediately seen that $L(T)(\theta)$ has the form (2). Since the zeros of the polynomial $\sum_{\nu=0}^{m} z^\nu$ lie on the circle $|z| = 1$, according to Corollary 3 the zeros of the Riemann sums

$$T_m(\theta) = \frac{(\lambda + \mu)i}{m} \sum_{\nu=1}^{m} T\left(\theta + \frac{\lambda i}{m}(m - \nu) - \frac{\mu\nu}{m} i\right)$$

lie in $\operatorname{Im} \theta \leqq 0$ and the conclusion follows from the Hurwitz theorem.

THEOREM 6. *If the conditions of Theorem* 4 *are satisfied, the inequality* $|L(S)(\theta)| < |L(T)(\theta)|$, $\operatorname{Im} \theta > 0$, *where the operator L is given by* (11), *holds.*

PROOF. Let $\mathfrak{M}$ be the complex linear space of trigonometric polynomials of degree not exceeding $n$, and $\mathfrak{N} = P_n$. According to Theorem 5 we have $L(\mathfrak{N}) \subset \mathfrak{N}$ and we complete the proof by applying Theorem 1.

Now we need the following

DEFINITION. A trigonometric polynomial of the form

$$T(\theta) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu\theta}, \qquad a_n a_{-n} \neq 0,$$

will be called balanced.

The following theorem is analogous to Theorem 2.

THEOREM 7. *Let* $T(\theta)$ *be a balanced trigonometric polynomial with zeros in the strip* $a \leqq \operatorname{Im} \theta \leqq b$. *Then all the zeros of* $N(\theta) = T(\theta + \lambda i) - \gamma T(\theta - \lambda i)$, $|\gamma| = 1$, $\lambda > 0$, *also lie in this strip.*

PROOF. Obviously $T(\theta + bi)$ belongs to $P_n$. Applying Theorem 4 to the pair $S(\theta) \equiv 0$ and $T(\theta + bi)$, we conclude that $N(\theta) \neq 0$ in $\operatorname{Im} \theta > b$. Since $T(-\theta + ai) \in P_n$, by the same reasoning $N(\theta)$ has no zeros in $\operatorname{Im} \theta < a$.

COROLLARY 1. (*Gauss-Lucas theorem for trigonometric polynomials.*) *If* $T(\theta)$ *is balanced and has zeros only in the strip* $\Omega$: $a \leqq \operatorname{Im} \theta \leqq b$, *the zeros of its derivative* $T'(\theta)$ *also lie in* $\Omega$.

PROOF. Letting $\lambda \to 0$ in $(T(\theta + \lambda i) - T(\theta - \lambda i))/\lambda$ and applying Hurwitz's theorem, we obtain the proof.

Going into details, one could prove that $T'(\theta)$ may have a zero $\theta_0$ on one of the lines $\operatorname{Im} \theta = a$, $\operatorname{Im} \theta = b$, such that $T(\theta_0) \neq 0$, if and only if all the zeros of $T(\theta)$ lie on the same line [5].

COROLLARY 2. *If* $T(\theta)$ *is balanced and has zeros only in the strip* $\Omega$, *then all the zeros of the trigonometric polynomial*

$$L(T)(\theta) = \int_{\theta-\lambda i}^{\theta+\lambda i} T(t)\,dt, \qquad \lambda > 0,$$

*also lie in* $\Omega$.

PROOF. This corollary can be deduced from Theorem 7 exactly as Theorem 5 was deduced from Corollary 3 of Theorem 4.

3. It is obvious that Theorem 4, with $|\tau| \leqq 1$, could be extended to the case when $S(\theta)$ and $T(\theta)$ are entire functions of exponential type belonging to appropriate classes. Here, for the sake of brevity, we shall confine ourselves to deriving some consequences of Theorem 4 concerning entire functions of exponential type bounded on the real axis, including Boas' inequality.

Let $f(z)$ be an entire function of exponential type $\sigma$, subject to the condition $|f(z)| \leqq M$ on the real axis. According to a theorem of B. Levitan [7, p. 193], there exists a sequence of trigonometric polynomials $S_n(z) = \sum_{\nu=-n}^{n} a_{\nu,n} e^{-i\nu\sigma z/n}$, tending uniformly to $f(z)$ on every bounded set, and such that the inequality $|S_n(z)| \leqq M$ is satisfied on the real axis. Applying (9) to the pair $S_n((n/\sigma)\theta)$ and $Me^{-in\theta}$, we obtain

$$\begin{aligned}(12)\qquad &\left|e^{-iw} S_n\left(\frac{n}{\sigma}(\theta_n + \lambda_n i)\right) + e^{iw} S_n\left(\frac{n}{\sigma}(\theta_n - \mu_n i)\right)\right| \\ &\leqq M|e^{-in\theta_n}|\,|e^{-iw+n\lambda_n} + e^{iw-n\mu_n}|\end{aligned}$$

where $\theta_n = \sigma z/n$, $\operatorname{Im} z \geqq 0$, $\lambda_n = \sigma\lambda/n$, $\mu_n = \sigma\mu/n$, $0 \leqq \mu \leqq \lambda$, $\lambda > 0$.

Letting $n \to \infty$ in (12) we get

$$\begin{aligned}(13)\qquad &|e^{-iw} f(z + \lambda i) + e^{iw} f(z - \mu i)| \\ &\leqq M|e^{-iw} e^{-i\sigma(z+\lambda i)} + e^{iw} e^{-i\sigma(z-\mu i)}|, \qquad 0 \leqq \mu \leqq \lambda,\ \operatorname{Im} z \geqq 0,\end{aligned}$$

in which Boas' inequality is included.

Finally, applying (13) twice with $w = \pi/2$, $z$ real, $\lambda = |y|$, $\mu = 0$ to the functions $f(z)$ and $f(-z)$, we obtain

$$(14)\qquad |f(x + iy) - f(x)| \leqq M(e^{\sigma|y|} - 1), \qquad x, y \text{ real},$$

from which Bernstein's inequality follows again.

In the same way we deduce from Theorem 6 the inequality

$$(15)\qquad \left|\int_{z-\mu i}^{z+\lambda i} f(t)\,dt\right| \leqq M\left|\int_{z-\mu i}^{z+\lambda i} e^{-i\sigma t}\,dt\right| \leqq \frac{M}{\sigma}|e^{\sigma(y+\lambda)} - e^{\sigma(y-\mu)}|$$

where $0 \leqq \mu \leqq \lambda$, $y = \operatorname{Im} z \geqq 0$, and of course, $|f(z)| \leqq M$ on the real axis.

The inequalities (14) and (15) are obviously exact.

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