

ON THE STRAIGHTNESS OF REDUCED TEICHMÜLLER SPACE

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ABSTRACT. Under a natural injection which is shown to be an isometry, the image of the reduced Teichmüller space $T^\#(G)$ in the open straight Teichmüller space of the Fuchsian model of G , $T(H)$, is an open straight subspace.

It is known that, with the Teichmüller metric, the Teichmüller space $T(H)$ of a nonelementary finitely generated Fuchsian group H of the first kind is an open straight space in the sense of Busemann [9], [10]. Hence, between any two distinct points of $T(H)$ there is a unique geodesic locally isometric to \mathbf{R} . This paper considers an extension of this property to $T^\#(G)$, the reduced Teichmüller space of a finitely generated nonelementary Fuchsian group G of the second kind.

Let $M_1(H)$ ($M_1(G)$) be the set of Beltrami differentials $\mu(z)d\bar{z}/dz$ on the upper half plane U satisfying $\|\mu\| = \text{ess sup} |\mu(z)| < 1$, $\mu(h(z))\overline{h'(z)}/h'(z) = \mu(z)$ for all $h \in H$ ($g \in G$).

Extend $\mu(z)$ to \mathbf{C} by $\mu(\bar{z}) = \bar{\mu}(z)$, and let $w_\mu(z)$ be the unique solution of the Beltrami differential equation $w_{\bar{z}} = \mu w_z$ fixing 0, 1, and ∞ . The Teichmüller space $T(H)$ ($T(G)$) is the set of equivalence classes $[w_\mu]$ of elements of $M_1(H)$ ($M_1(G)$) where $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on \mathbf{R} . The reduced Teichmüller space is the set of equivalence classes θ_μ of elements of $M_1(G)$ where $\mu \sim_G \nu$ if and only if $w_\mu = w_\nu$ on $\Lambda(G)$, the limit set of G . This is in turn equivalent to the condition that the induced isomorphisms $g \rightarrow w_\mu g w_\mu^{-1}$ and $g \rightarrow w_\nu g w_\nu^{-1}$ are identical. (Note that for H of the first kind $T^\#(H) = T(H)$.)

Since G is a finitely generated Fuchsian group of the second kind, $\Lambda(G) \subsetneq \mathbf{R}$, and we have $U \xrightarrow{\rho} \Omega(G) \xrightarrow{\pi} \Omega(G)/G$ where ρ is a holomorphic cover map, π is a (possibly ramified) holomorphic cover, $\Omega(G)$ is the ordinary set of G , and $\Omega(G)/G$ is the double of U/G . Let $J: U \rightarrow U$ be given by $J(z) = -\bar{z}$; the cover ρ may be chosen to satisfy $\rho(J(z)) = \bar{\rho}(z)$. Fix H by defining $H = \{h \in \text{PSL}(2, \mathbf{R}) | \rho \circ h = g \circ \rho \text{ for some } g \in G\}$.

Let $B_2(H, U)$ be the set of quadratic differentials with respect to H on U satisfying $\|\phi\| = \sup |\phi(z)y^2| < \infty$, and let $B'_2(H, U) = \{\phi \in B_2(H, U) | \phi(Jz) = \bar{\phi}(z)\}$. Let $B_2(G, \Omega)$ be those quadratic differentials on $\Omega(G)$ which are real

Received by the editors February 21, 1975.

AMS (MOS) subject classifications (1970). Primary 32G15.

Key words and phrases. Teichmüller space, reduced Teichmüller space, straightness, Beltrami differential, quadratic differential, Teichmüller mapping.

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on $\Omega(G) \cap \mathbf{R}$ and satisfy $\|\psi\| = \sup|(\psi \cdot \lambda^{-2})(z)| < \infty$, where λ is the Poincaré metric induced by $\pi \circ \rho$.

Finally, let $M'_1(H) = \{\eta \in M_1(H) | \eta(J(z)) = \bar{\eta}(z)\}$; this is equivalent to the condition that w_η commutes with J . Via the isometries $\mu(z) \rightarrow \rho \cdot \mu(z) = \mu(\rho(z))(\rho'(z)/\rho'(z))$ and $\psi(z) \rightarrow (\psi \times \rho)(z) = \psi(\rho(z))(\rho'(z))^2$, it can be easily demonstrated that $M_1(G)$ is isomorphic to $M'_1(H)$, and $B_2(G, \Omega)$ is isomorphic to $B'_2(H, U)$. Earle has shown [7], [8] that the former induces the map $\theta_\mu \rightarrow [w_{\rho \cdot \mu}]$ which is a real analytic injection of $T^*(G)$ into $T(H)$ with image $T'(H) = \{[w_\eta] \in T(H) | \eta \in M'_1(H) \text{ for some } \eta \sim \hat{\eta}\}$.

1. The straightness of $T'(H)$. We wish to study further the structure of $T'(H)$. It is clear that if $\mu \in [w_\mu] \in T'(H)$, w_μ commutes with J on \mathbf{R} (i.e., is odd).

LEMMA 1. *The unique extremal element w_η in each equivalence class $[w_\eta] \in T(H)$ with odd boundary values is symmetric ($w_\eta \circ J(z) = J \circ w_\eta(z)$ on U), and $T'(H) = \{[w_\mu] \in T(H) | w_\mu \text{ is odd on } \mathbf{R}\}$.*

PROOF. Suppose $w = w_\eta$ is not symmetric, and set $f(z) = (J \circ w \circ J)(z)$. Then the complex dilatation η_1 of f is $(\bar{\eta} \circ J)(z)$. Since J is a homeomorphism, $\|\eta_1\| = \|\bar{\eta} \circ J\|$. Since w_η is odd on \mathbf{R} , $f(x) = w_\eta(x)$, and $f \sim w_\eta$. Since $\|\eta_1\| = \|\eta\|$, by Teichmüller's theorem for $T(H)$, $J \circ w_\eta \circ J = f = w_\eta$, and so w_η commutes with J on U . Hence $\eta \in M'_1(H)$, and equivalence classes of $T(H)$ with odd boundary values are contained in $T'(H)$. The other inclusion is obvious. \square

Let w_η be the extremal element of $[w_\eta] \in T'(H)$. Since $M(G)$ is isomorphic to $M'(H)$, $w_\eta = w_{\rho \cdot \alpha}$ for some $\alpha \in M_1(G)$. Now α must be uniquely extremal, for if $\beta \in M_1(G)$, $\alpha \sim \beta$, and $\|\beta\| \leq \|\alpha\|$, then $\rho \cdot \beta \sim \rho \cdot \alpha$ with $\|\rho \cdot \beta\| \leq \|\rho \cdot \alpha\| = \|\eta\|$, which is a contradiction. Hence

PROPOSITION 1. *Each equivalence class θ_μ of $T^*(G)$ contains a unique extremal element w_μ ; $w_{\rho \cdot \mu}$ is the extremal element of $[w_{\rho \cdot \mu}] \in T'(H)$.*

Since we now have an extremal element in each class, the Teichmüller metric may be defined on $T^*(G)$.

The set of fixed points of an involutonic isometry of an open straight space is a nonempty open straight space [6]. We show $T'(H)$ is such a set.

LEMMA 2. *Let $h \in H$. Then $h \circ J = J \circ h_0$ for some $h_0 \in H$.*

PROOF. Let $h(z) = (az + b)/(cz + d) \in PSL(2, \mathbf{R})$. Then $h \circ J(z) = J \circ m(z)$ where $m(z) = (az - b)/(-cz + d)$ and $m \in PSL(2, \mathbf{R})$. Since $\rho \circ J = \bar{\rho}$, by the definition of H , $\rho \circ m(z) = g \circ \rho(z)$ for some $g \in G$. Hence $m = h_0 \in H$, and consequently $JHJ^{-1} = H$. Note also that $h_z \circ J(z) = (\bar{h}_0)_z(z)$. \square

Let $\gamma \in M_1(H)$. Then $\bar{\gamma} \circ J \in M_1(H)$. As in the proof of Lemma 1, if w_γ has complex dilatation γ , the map $f(z) = (J \circ w_\gamma \circ J)(z)$ has complex dilatation $\bar{\gamma} \circ J$. If $w_\eta = w_\gamma$ on \mathbf{R} , $w_{\bar{\gamma} \circ J} = m \circ J \circ w_\gamma \circ J = m \circ J \circ w_\eta \circ J = w_{\bar{\eta} \circ J}$ on \mathbf{R} , where $m \in PSL(2, \mathbf{R})$ is the normalizer. Thus the map J induces the well-defined map $J^* = T(H) \rightarrow T(H)$ by $[w_\gamma] \rightarrow [w_{\bar{\gamma} \circ J}]$. J^* is clearly an involution since $\overline{(\bar{\gamma} \circ J)} \circ J = \gamma$, and it is also an isometry for

$$\left| \frac{\bar{\gamma} \circ J - \bar{\eta} \circ J}{1 - (\eta\bar{\gamma}) \circ J} \right| = \left| \frac{\bar{\gamma} - \bar{\eta}}{1 - \eta\bar{\gamma}} \right| \circ J = \left| \frac{\gamma - \eta}{1 - \bar{\eta}\gamma} \right| \circ J.$$

Since J is a homeomorphism, the essential supremums over U are equal. Hence J^* is (at most) distance decreasing. However, since $(J^*)^2$ is the identity, J must be an isometry.

PROPOSITION 2. $T'(H)$ is an open straight space.

PROOF. We show that $T'(H)$ is the set of fixed points of the isometric involution J^* of $T(H)$.

The equivalence class $[w_{\bar{\eta}}] \in T'(H)$ if and only if it contains an element w_{η} whose dilatation satisfies $\eta \circ J = \bar{\eta}$, or equivalently, $\bar{\eta} \circ J = \eta$. Each point of $T'(H)$ is thus clearly fixed by J^* .

Suppose conversely that $J^*([w_{\hat{\gamma}}]) = [w_{\hat{\gamma}}]$. Then $\bar{\hat{\gamma}} \circ J = \gamma' \sim \hat{\gamma}$. In particular, $[w_{\hat{\gamma}}]$ contains a Teichmüller mapping $w_{\gamma'}$, and as such, $w_{\gamma'}$ has the strictly smallest dilatation. But $\|\gamma'\| = \|\bar{\gamma} \cdot J\|$ which implies $[w_{\hat{\gamma}}] = [w_{\gamma'}] \in T'(H)$. Thus $T'(H)$ is the fixed point set of J^* . \square

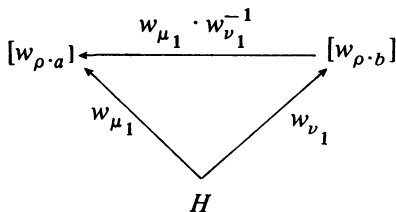
2. The straightness of $T^*(G)$. We wish now to transfer the straight space structure of $T'(H)$ to $T^*(G)$. The map $T^*(G) \rightarrow T'(H)$ given by $\theta_{\mu} \rightarrow [w_{\rho \cdot \mu}]$ is one-to-one and onto. Suppose $w_{\rho \cdot \mu}$, $w_{\rho \cdot \nu}$ belong to $[w_{\rho \cdot a}]$, $[w_{\rho \cdot b}]$ respectively. Set $k(w_{\rho \cdot \mu} \circ w_{\rho \cdot \nu}^{-1}) = \text{dilatation of } w_{\rho \cdot \mu} \circ w_{\rho \cdot \nu}^{-1}$. Then

$$k(w_{\rho \cdot \mu} \circ w_{\rho \cdot \nu}^{-1}) = \|(\mu - \nu) / (1 - \bar{\nu}\mu)\|$$

since ρ is onto. Thus

$$d^*(\theta_a, \theta_b) \geq d([w_{\rho \cdot a}], [w_{\rho \cdot b}]),$$

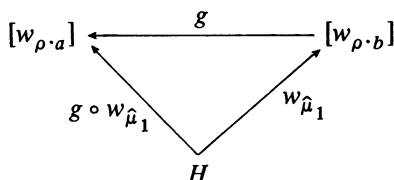
since the isomorphism $\theta_{\rho \cdot a}$, $\theta_{\rho \cdot b}$ may be induced by maps which are odd on \mathbf{R} but do not commute with J on U . Consider



Suppose $k(w_{\mu_1} \circ w_{\nu_1}^{-1}) < k(w_{\rho \cdot \mu} \circ w_{\rho \cdot \nu}^{-1})$ for all $\rho \cdot \mu$, $\rho \cdot \nu \in M'_1(H)$ inducing $\theta_{\rho \cdot a}$, $\theta_{\rho \cdot b}$ respectively, where either μ_1 or $\nu_1 \notin M'_1(H)$.

Replace w_{ν_1} by $w_{\hat{\mu}_1} \sim w_{\nu_1}$ where $w_{\hat{\mu}_1}$ commutes with J . By Lemma 1, since $w_{\mu_1} \circ w_{\nu_1}^{-1}$ is odd on \mathbf{R} we may replace it by an equivalent map g which satisfies $k(g) \leq k(w_{\mu_1} \circ w_{\nu_1}^{-1})$ and $g \circ J = J \circ g$ on U .

The diagram is then modified to



where $g \circ w_{\hat{\mu}_1} \sim w_{\mu_1}$ and $g \circ w_{\hat{\mu}_1}$ is symmetric. Then $k((g \circ w_{\hat{\mu}_1}) \circ w_{\hat{\mu}_1}^{-1}) = k(g) \leq k(w_{\mu_1} \circ w_{\nu_1}^{-1})$, a contradiction. The distance d will therefore be determined by Beltrami differentials in $M'_1(H)$, and the inequality is actually equality. The map $T^*(G) \rightarrow T'(H)$ (and its inverse) is an isometry, and $T^*(G)$ is an open straight space. We have proved

THEOREM. *Let G be a normalized finitely generated Fuchsian group. Then $T^*(G)$ is an open straight space.*

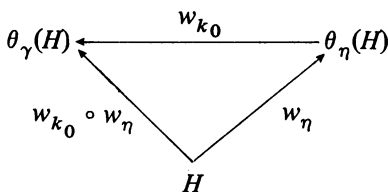
3. A second characterization of $T'(H)$. The equivalence class $[w_\eta] \in T(H)$ is contained in $T'(H)$ if and only if θ_η is induced by some w_η satisfying $w_\eta \circ J = J \circ w_\eta$. Let $\phi(z)dz^2 \in B'_2(H_\eta, U)$. The ray in $T(H)$ through $[w_\eta]$ determined by ϕ lies entirely in $T'(H)$, for set $\eta_a(z) = a\bar{\phi}(z)/|\phi(z)|$, $0 \leq a < 1$. Then $\eta_a(Jz) = a\bar{\phi}(Jz)/|\phi(Jz)| = a\phi(z)/|\bar{\phi}(z)| = \bar{\eta}_a(z)$ and $\eta_a \in M'_1(H_\eta)$ for $0 \leq a < 1$. Hence w_{η_a} commutes with J , $w_{\eta_a} \circ w_\eta$ commutes with J , and $[w_{\eta_a} \circ w_\eta] \in T'(H)$.

On the other hand, suppose $[w_\eta], [w_\gamma] \in T'(H)$. Then w_η, w_γ are odd one-to-one maps of \mathbf{R} onto itself. Since $[w_\eta], [w_\gamma]$ are also in $T(H)$, they lie on a unique straight line L in $T(H)$. In particular, they lie on the ray R of L determined by $\phi(z)dz^2$, $\phi \in B_2(H_\eta, U)$ where $k_0\bar{\phi}(z)/|\phi(z)|$, $0 \leq k_0 < 1$, is the dilatation of the (unique, extremal) Teichmüller map from $[w_\eta]$ to $[w_\gamma]$. We wish to show $\phi(z)dz^2 \in B'_2(H_\eta, U)$, for then this ray would lie in $T'(H)$. The same argument beginning with the inverse map $[w_\gamma] \rightarrow [w_\eta]$ will show that all of L lies in $T'(H)$.

The points of R are determined by maps w_k where

$$(w_k)_z(z) = k \frac{\bar{\phi}(z)}{|\phi(z)|} (w_k)_z(z), \quad 0 \leq k < 1, \phi \in B_2(H_\eta, U).$$

We assume w_η determines $\theta_\eta(H)$, and w_{k_0} determines $\theta_\gamma(H) = \theta_{k_0} \circ \theta_\eta(H)$.



Now $w_{k_0} \circ w_\eta$ and w_γ determine the same point in $T(H)$, hence they agree on \mathbf{R} . Thus

$$w_{k_0}(-x) = w_\gamma \circ w_\eta^{-1}(-x) = -(w_\gamma \circ w_\eta^{-1})(x) = -w_{k_0}(x)$$

and w_{k_0} is odd on \mathbf{R} . Since w_{k_0} is uniquely extremal, by Lemma 1 it must be symmetric with respect to J .

For a symmetric w , $(w_\gamma \circ J) = \bar{w}_z$, and $(w_\gamma \circ J) = \bar{w}_z$. For $w = w_{k_0}$, if $w_z(z) = k_0\bar{\phi}(z)w_z(z)/|\phi(z)|$, then

$$(w_z \circ J)(z) = k_0 \frac{(\bar{\phi} \circ J)(z)}{|(\bar{\phi} \circ J)(z)|} (w_z \circ J)(z), \quad \bar{w}_z(z) = k_0 \frac{(\bar{\phi} \circ J)(z)}{|(\bar{\phi} \circ J)(z)|} \bar{w}_z(z)$$

and consequently,

$$\frac{(\bar{\phi} \circ J)(z)}{|(\phi \circ J)(z)|} = \frac{\phi(z)}{|\bar{\phi}(z)|}.$$

Except for isolated zeros of ϕ , the function $(\bar{\phi} \circ J)(z)/\phi(z) = |(\phi \circ J)(z)/\bar{\phi}(z)|$ is conformal with constant zero imaginary part, and hence is constant. Thus $(\bar{\phi} \circ J)(z) = c\phi(z)$ for some $c \in \mathbf{R}$, $c > 0$. On the y axis, $J(y) = y$, and so $\bar{\phi}(y) = c\phi(y)$. Hence $c = 1$, $\phi \circ J = \bar{\phi}$, and $\phi(z)dz^2 \in B'_2(H_\eta, U)$. We have proved

PROPOSITION 3. *The straight line connecting two points of $T'(H)$ lies in $T'(H)$, and is determined by symmetric quadratic differentials $(\phi \circ J = \bar{\phi})$.*

$T'(H)$ is a metric space when the Teichmüller metric is restricted to it. Let $\{\theta_n(H)\}$ be a bounded sequence of points of $T'(H)$. Since $\{\theta_n(H)\} \subset T(H)$, it must contain a subsequence converging to $\theta(H) \in T(H)$. But $\theta_n(H) \rightarrow \theta(H)$ if and only if the sequence of boundary values $\{w_n|_{\mathbf{R}}\} \rightarrow w|_{\mathbf{R}}$. By Lemma 1, $T'(H)$ is the set of equivalence classes of $T(H)$ with odd boundary values. Thus $[w] = \theta(H)$ must also be in $T'(H)$, and $T'(H)$ is finitely compact. The unique straight line in $T(H)$ connecting two points of $T'(H)$ lies in $T'(H)$; a second geodesic connecting two points of $T'(H)$ would also be one for $T(H)$, contradicting the uniqueness there. We have again proved

PROPOSITION 2'. *$T'(H)$ is an open straight space.*

Proposition 3 also allows us to complete the proof of Teichmüller's theorem for $T^\#(G)$.

PROPOSITION 4. *The extremal element of θ_μ is a Teichmüller mapping.*

PROOF. Let w_μ be the unique extremal element of $\theta_\mu \in T^\#(G)$. Then $\rho \cdot \mu \in M_1(H)$ is by Proposition 1, the dilatation of the unique extremal element of $[w_{\rho\mu}]$, which is symmetric with respect to J . By Teichmüller's theorem for $T(H)$, $(\rho \cdot \mu)(z) = k\bar{\phi}(z)/|\phi(z)|$ for some k , $0 \leq k < 1$, $\phi \in B_2(H, U)$. But ϕ determines a ray between two points of $T'(H)$ (from $\theta_{\text{id}}(H) = (H)$ to $\theta_{\rho\mu}(H)$) and therefore belongs to $B'_2(H, U)$ by Proposition 3. But $\phi = \psi \times \rho$ where $\psi \in B_2(G, \Omega)$, and

$$(\rho \cdot \mu)(z) = k \frac{\bar{\phi}(z)}{|\phi(z)|} = k \frac{(\overline{\psi \times \rho})(z)}{|(\psi \times \rho)(z)|} = \rho \cdot \left(k \frac{\bar{\psi}}{|\psi|} \right)(z)$$

which implies that $\mu = k\bar{\psi}/|\psi|$. Since $\psi \in B_2(G, \Omega)$, w_μ is a Teichmüller mapping, and we have completed the proof of Teichmüller's theorem for finitely generated normalized Fuchsian groups of the second kind.

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