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CONFIGURATION-LIKE SPACES AND THE BORSUK-ULAM THEOREM

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ABSTRACT. Some extensions of the classical Borsuk-Ulam Theorem are proved by computing a bound on the homology of certain spaces similar to configuration spaces. The Bourgin-Yang Theorem and a generalization due to Munkholm are special cases of these results.

1. **Introduction.** The purpose of this paper is to extend and unify several generalizations of the Borsuk-Ulam Theorem. Let π_p denote the cyclic group of prime order p and let X be a pathwise connected Hausdorff space on which π_p acts freely. Suppose that M is some fixed manifold and that $f\colon X \to M$ is any map. We are interested in conditions on X, depending on M but not on f, which are sufficient to insure that a certain number of points in some orbit are sent to the same point in M by f. Specifically, let σ denote the generator of π_p and define

$$A(f,q) = \{x \in X \mid \text{there exist } i_1, i_2, \ldots, i_q \text{ with } 0 \leqq i_1 < i_2 < \cdots < i_q < p \text{ and } f(\sigma^{i_1}x) = f(\sigma^{i_2}x) = \cdots = f(\sigma^{i_q}x)\}.$$

In the case $M = R^n$, we prove the following, in which $\dim A$ denotes the covering dimension of A, and all cohomology is taken with $\mathbf{Z}_p$ coefficients unless otherwise stated.

THEOREM 1. *If $H^i(X) = 0$ for $0 < i < (n-1)(p-1) + q - 1$ and $q \geqq \frac{1}{2}(p+1)$ or $q = 2$, then $A(f,q) \neq \varnothing$.*

THEOREM 2. *If X is a $\mathbf{Z}_p$-orientable m-manifold and $H^i(X) = 0$ for $0 < i < (n-1)(p-1) + q - 1$ and $q \geqq \frac{1}{2}(p+1)$ or $q = 2$, then $\dim A \geqq m - (n-1)(p-1) - q + 1$.*

Special cases of these theorems are known:

1. The classical Borsuk-Ulam Theorem is Theorem 1 with $X = S^n$ and $q = p = 2$ [1].

2. The "mod p Bourgin-Yang Theorem" of Munkholm is Theorem 2 with $q = p$ and X a mod p homology m-sphere [6]. For this special case the proof given below is much simpler than Munkholm's.

3. The case $q = 2$ of Theorem 1 appears in [3].

Theorems 1 and 2 are actually special cases of a more general theorem, in which R^n is replaced by an arbitrary manifold M. That is, for each M and

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$q \leqq p$ there is a number $N(M,p,q)$ (defined below) such that for $f\colon X \to M$ we have:

THEOREM 3. *If $H^i(X) = 0$ for $0 < i \leqq N(M,p,q)$, then $A(f,q) \neq \varnothing$. If in addition we assume that X is a $\mathbf{Z}_p$-orientable m-manifold, then* $\dim A(f,q) \geqq m - N(M,p,q) - 1$.

To define the numbers $N(M,p,q)$, consider the subspace $G(M,p,q)$ of $(M)^p$ consisting of the p-tuples in which no q coordinates coincide. More precisely,

$$G(M,p,q) = \{(x_1,\dots,x_p) | \text{for any } \{x_{i_1},\dots,x_{i_q}\} \text{ with } 0 < i_1 < \dots < i_q \leqq p, \text{ at least 2 of the } x_{i_j}\text{'s are different}\}.$$

Note that $G(M,p,q) \subset G(M,p,q+1)$, $G(M,p,p) = (M)^p - \Delta_M$, and $G(M,p,2)$ is the Fadell-Neuwirth configuration space [5]. The group π_p acts freely on $G(M,p,q)$ by cyclic permutation of coordinates, and the inclusions $G(M,p,q) \subset G(M,p,p)$ are equivariant. For some large n, $G(M,p,q)$ embeds in $G(R^n,p,q)$ via the embedding of M in R^n. Define

$$G(R^\infty,p,q) = \varinjlim_n G(R^n,p,q).$$

PROPOSITION. *$G(R^\infty,p,p)$ is a free π_p-space with trivial homotopy groups and hence $G(R^\infty,p,p)/\pi_p$ is a $K(\pi_p,1)$.*

PROOF. Since

$$H_*(G(R^\infty,p,p);\mathbf{Z}) = H_*(\varinjlim G(R^n,p,p);\mathbf{Z}) = \varinjlim H_*(G(R^n,p,p);\mathbf{Z})$$

and $G(R^n,p,p) \simeq S^{n(p-1)-1}$, $G(R^\infty,p,p)$ has trivial homology groups. Since $G(R^\infty,p,p)$ is simply connected, the result follows from the Hurewicz Theorem.

DEFINITION OF $N(M,p,q)$. Let ϕ be an equivariant embedding of $G(M,p,q)$ in $G(R^\infty,p,p)$. Recall that $H^iK(\pi_p,1) = \mathbf{Z}_p$ for all i and define $N(M,p,q)$ to be the largest N such that ϕ^* is not the zero homomorphism. We have not calculated $N(M,p,q)$ for $M \neq R^n$ except for the case $q = 2$ (see [4]). When $M = R^n$, it is sufficient to calculate the first nonvanishing homology class in a certain union of spheres. This we do in §3. The result is:

THEOREM 4. *$N(R^n,p,q) \leqq (n-1)(p-1) + q - 2$ if $q \geqq \frac{1}{2}(p+1)$ or $q = 2$.*

2. **Proofs of Theorems 1, 2, and 3.** We prove Theorem 3. Theorems 1 and 2 follow immediately from Theorems 3 and 4. Let σ be the generator of π_p and define $\psi\colon X \to (M)^p$ by $\psi(x) = (f(x), f(\sigma x), \dots, f(\sigma^{p-1}x))$. If $A(f,q) = \varnothing$ then ψ is an equivariant map of X into $G(M,p,q)$. Consider the following diagram, in which the vertical arrows represent projections.

$$\begin{array}{ccccc} X & \xrightarrow{\psi} & G(M,p,q) & \xrightarrow{\phi} & G(R^\infty,p,p) \\ \downarrow & & \downarrow & & \downarrow \\ X/\pi_p & \xrightarrow{\hat{\psi}} & G(M,p,q)/\pi_p & \xrightarrow{\hat{\phi}} & G(R^\infty,p,p)/\pi_p \end{array}$$

If $H^i(X) = 0$ for $0 < i \leqq N(M,p,q)$, then it follows from the naturality of the spectral sequence for a covering that $(\hat{\phi}\hat{\psi})^*$ is a monomorphism in degrees less than or equal to $N(M,p,q) + 1$, contradicting the fact that $\hat{\phi}^* = 0$ in degrees greater than $N(M,p,q)$. This proves the first part of the theorem.

Now suppose that X is a $\mathbf{Z}_p$-orientable m-manifold. Observe that ψ restricts to an equivariant map of $X - A(f,q)$ into $G(M,p,q)$ and that we may assume $X - A(f,q)$ is path connected. By the above argument there must be some j, $0 < j \leqq N(M,p,q)$, such that $H^j(X - A(f,q)) \neq 0$, and hence

$$H_j(X - A(f,q)) \neq 0.$$

By Alexander Duality, $H^{m-j}(X, A(f,q)) \neq 0$. Similarly $H_j(X) = 0$ implies $H^{m-j}(X) = 0$, so by the exact cohomology sequence

$$\overline{H}^{m-j-1}(A(f,q)) \neq 0.$$

By the argument which appears in [6], this is enough to prove that the covering dimension of $A(f,q)$ is greater than or equal to $m - N(M,p,q) - 1$.

3. **Proof of Theorem 4.** First we remark that the case $p = q$ is particularly simple since $G(R^n,p,p) = (R^n)^p - \Delta \simeq S^{n(p-1)-1}$, and so $N(R^n,p,p) \leqq n(p-1) - 1$. The case $q = 2$ appears in [4]. In general, we proceed as follows. The standard strong deformation retraction of $R^{np} - \{0\}$ onto S^{np-1} restricts to a strong deformation retraction of $G(R^n,p,q)$ onto its intersection with S^{np-1}. Let $K(n,p,p-q)$ denote the complement of the image of $G(R^n,p,q)$ under this deformation. We let $k = p - q$ and note that $K(n,p,k)$ is the union of spheres of dimension $n(k+1) - 1$. Our method of bounding $N(R^n,p,q)$ will be the rather crude one of bounding $H^* G(R^n,p,q)/\mathbf{Z}_p$. In general we will do this by finding a lower bound for $H_* K(n,p,k)$ using the Mayer-Vietoris sequence and then applying Alexander Duality in the $(np - 1)$-sphere.

First we need some notation for the pieces of $K(n,p,k)$ to which we will apply Mayer-Vietoris. Let $I = (i_1, \ldots, i_j)$, $j \leqq k$, denote any j-tuple of integers with $0 < i_1 < i_2 < \cdots < i_j \leqq p$. We define the *length* of I to be j and denote it by $l(I)$. We also permit I to be empty and in this case define $l(I) = 0$. Now let m be any positive integer less than or equal to $p - k$ and define

$$\begin{aligned} W(I,k,m) = \{(x,x,\ldots,x,y_1,x,x,\ldots,x,y_2,x,x,\ldots,x,y_k,x,x,\ldots,x) |\ & x \in R^n, y_s \text{ occurs in the } i_s \text{ th place for } s = 1, 2, \ldots, j, \\ & \text{and there are } mx\text{'s between } y_j \text{ and } y_{j+1}\}. \end{aligned}$$

That is, the coordinates which are not specified to be equal to other coordinates occur in places $i_1, i_2, \ldots, i_j, i_j + m + 1$, and beyond. By abuse of notation we write the sequence $x, x, \ldots, x$ (α_1 terms) as x^{α_1}. A typical point in $W(I,k,m)$ looks like

$$(x^{\alpha_1} y_1 x^{\alpha_2} y_2 \cdots x^{\alpha_j} y_j x^m y_{j+1} x^{l_1} y_{j+2} x^{l_2} \cdots y_k x^{l_{k-j}}),$$

where $\alpha_1 + \cdots + \alpha_j + m + l_1 + \cdots + l_{k-j} = p - k = q$. We note that the

α_i's are determined by I and ignore them. Observe that $W(I,k,m)$ is a union of equatorial $(n(k+1)-1)$-spheres in S^{np-1}. We assume that $q \geqq \frac{1}{2}(p+1)$.

LEMMA 1. $[\bigcup_{i=0}^{m} W(I,k,i)] \cap W(I,k,m+1) = W(I,k-1,m+1)$.

PROOF. Observe that a point is in the left-hand side if and only if $y_{j+1} = x$. Therefore $y_{j+2}, \ldots, y_k$ can be relabeled $y_{j+1}, \ldots, y_{k-1}$.

LEMMA 2. $H_i W(I,k,m) = 0$ *if* $0 < i < n+k-1$.

PROOF. The proof is by induction on k and for fixed k by downward induction on $l(I)$. The lemma is true for $k = 0$ since $W(I,0,m)$ is an $(n-1)$-sphere. Fix k and assume that the lemma is true with $k-1$ replacing k. The induction on $l(I)$ starts with $l(I) = k$. In this case $W(I,k,m)$ is an $(n(k+1)-1)$-sphere, so the lemma is true. Now suppose that $l(I) = R-1$ and that the lemma is true for all I with $k \geqq l(I) \geqq R$. A point

$$x^{\alpha_1} y_1 \cdots y_{R-1} x^m (y_R x^{l_1} \cdots x^{l_{R-1-k}})$$

can be rewritten as

$$x^{\alpha_1} y_1 \cdots y_R x^{l_1} (y_{R+1} \cdots x^{l_{R-1-k}}),$$

so we have $W(I,k,m) = \bigcup_t W(J,k,t)$, where t varies from 0 to some number s determined by I, k, and m. Since $l(J) > l(I)$, $H_i W(J,k,r) = 0$ for $0 < i < n+k-1$ and all r by induction. Now we assume that $H_i(\bigcup_{t=0}^{r} W(J,k,t)) = 0$ for $0 < i < n+k-1$ and show that $H_i(\bigcup_{t=0}^{r+1} W(J,k,t)) = 0$ for $0 < i < n+k-1$. By Lemma 1 the Mayer-Vietoris sequence is

$$\cdots \to H_i\left(\bigcup_{t=0}^{r} W(J,k,t)\right) \oplus H_i W(J,k,r+1) \to H_i\left(\bigcup_{t=0}^{r+1} W(J,k,t)\right)$$
$$\to H_{i-1} W(J,k-1,r+1) \to \cdots$$

in which the left side is 0 for $0 < i < n+k-1$ by the inductions on r and on $l(I)$ and the right side is 0 for $0 < i < n+k-1$ by the induction on k. Therefore $H_i W(I,k,m) = H_i(W(J,k,t)) = 0$ for $0 < i < n+k-1$.

REMARK. Note that the second half of this proof shows that for any sequence J, if $H_i W(J,k,t) = 0$ for $0 < i < n+k-1$ and all t, then $H_i \bigcup_{t=0}^{s} W(J,k,t) = 0$ for $0 < i < n+k-1$, for any s.

PROOF OF THEOREM 4. First we note that $K(n,p,p-q) = \bigcup_{t=0}^{p} W(\varnothing,k,t)$, so by the above remark $H_i K(n,p,p-q) = 0$ for $0 < i < n+p-q-1$. By the remarks at the beginning of this section

$$H^i G(R^n,p,q) \cong H^i(S^{np-1} - K(n,p,p-q)),$$

which is in turn isomorphic to $H_{np-1-i}(S^{np-1}, K(n,p,p-q))$ by Alexander Duality. Then by the exact sequence

$$\cdots \to H_{np-1-i}(S^{np-1}) \to H_{np-1-i}(S^{np-1}, K(n,p,p-q))$$
$$\to H_{np-2-i}(K(n,p,p-q)) \to \cdots$$

we have that $H^i G(R^n,p,q) = 0$ for $i > (n-1)(p-1)+q-2$. This is sufficient, by the argument in [4], for example, to conclude that

$$H^i(G(R^n,p,q)/\mathbf{Z}_p) = 0 \quad \text{for } i > (n-1)(p-1) + q - 2.$$

4. **An example.** In some situations these results are best possible ones. For example, in [6] Munkholm gives an example for each odd p and each m of a π_p-action on $S^{m(p-1)-1}$ and a map from $S^{m(p-1)-1}$ to R^m such that no entire orbit is sent to the same point in R^m. Our Theorem 1 states that there is an orbit in which $p - 1$ points are sent to the same point. This example shows that in the case $q = p - 1$ one can have $H^i(X) = 0$ for $0 < i < (n-1) \cdot (p-1) + q - 1$ with $A(f, q+1) = \varnothing$.

REMARK. We conjecture that Theorem 4 and hence Theorems 1 and 2 are true without the restriction $q \geqq \frac{1}{2}(p+1)$, although it is not hard to see that Lemma 1 and hence our method of proof break down if $q < \frac{1}{2}(p+1)$. The difficulty can be seen in the case $I = \varnothing$, $p = 5$, $q = 2$, $m = 0$. Lemma 1 then says $W(\varnothing, 3, 0) \cap W(\varnothing, 3, 1) = W(\varnothing, 2, 1)$. However a point of the form (x, z, z, y, x) is in the left-hand side but not the right.

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