

PROJECTIVE MODULI OF CERTAIN QUOTIENT RINGS

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ABSTRACT. The author considers some properties of extension rings B of a ring A that satisfy the condition that every maximal ideal of B is an extension of some ideal of A . Such extensions have been used by D. Lissner, K. Lønsted, N. Moore, and A. Simis to obtain rings for which the projective moduli are arbitrarily less than the dimension of the maximal spectra. It is shown that families of prime ideals of maximal type can be used to construct such extensions.

Introduction. Throughout this paper all rings are assumed to be commutative with an identity and all extension rings are assumed to be unitary extensions. If B is an extension ring of a ring A , the pair (B, A) is said to have *Property E_M* if every maximal ideal of B can be written as IB for some ideal I of A .

We now list some results concerning pairs of rings with *Property E_M* . The proofs are elementary and are omitted.

RESULT 1.1. *If B is an extension ring of a ring A , then a necessary and a sufficient condition for (B, A) to have *Property E_M* is that $M = (M \cap A)B$ for each maximal ideal M of B .*

RESULT 1.2. *Suppose C is an extension ring of B and B is an extension ring of a ring A .*

(a) *If (C, A) has *Property E_M* , then (C, B) has *Property E_M* .*

(b) *If C is an integral extension of B and if both (C, B) and (B, A) have *Property E_M* , then (C, A) has *Property E_M* .*

RESULT 1.3. *If B is an extension ring of a ring A and (B, A) has *Property E_M* , then $[B/I, A/(I \cap A)]$ has *Property E_M* for each ideal I of B .*

For a ring A , $\max A$ denotes the maximal spectrum of A with the Zariski topology [1, Chapter III, §3].

THEOREM 1.4. *Suppose B is an extension ring of a ring A and (B, A) has *Property E_M* .*

(a) *If $m \in \max A$, then $mB = B$ or $mB \in \max B$.*

(b) *If $aB \neq B$ for all proper ideals a of A , then the mapping $M \rightarrow M \cap A$ defines a continuous bijection of $\max B$ into $\max A$. The inverse mapping is given by $m \rightarrow mB$.*

Received by the editors June 24, 1974.

AMS (MOS) subject classifications (1970). Primary 13B99; Secondary 13C10.

Key words and phrases. Projective modulus, families of prime ideals of maximal type, maximal spectrum, unimodular element, integral extension.

PROOF. Suppose $m \in \max A$ is such that $mB \neq B$. Then $mB \subseteq M$ for some maximal ideal M of B ; hence $m \subseteq M \cap A \neq A$. Therefore $m = M \cap A$ and $M = (M \cap A)B = mB$.

To prove condition (b), we let f be the mapping from $\max B$ defined by $f(M) = M \cap A$, and we let g be the mapping from $\max A$ defined by $g(m) = mB$. Condition (a) implies $g: \max A \rightarrow \max B$. We now show that $f: \max B \rightarrow \max A$. Suppose $M \in \max B$. Since $M \cap A \neq A$, there is a maximal ideal m of A which contains $M \cap A$. Then $M = (M \cap A)B \subseteq mB \neq B$; it follows that $M = mB$. Hence $M \cap A = m$. We have proved $M \cap A \in \max A$ and $gf(M) = M$. If $m \in \max A$, then $fg(m) = mB \cap A = m$. It is well known that if $h: A \rightarrow B$ is a ring homomorphism, then the map $\text{spec } B \rightarrow \text{spec } A$ given by $p \rightarrow h^{-1}(p)$ is continuous. By letting h be the inclusion mapping, we deduce that f is continuous.

PROPOSITION 1.5 [3], [6]. *If B is an integral extension of A such that for each maximal ideal m of A there is a unique ideal of B which lies over m , then $M \rightarrow M \cap A$ defines a homeomorphism from $\max B$ onto $\max A$.*

2. Unimodular elements. Throughout this section B is an extension ring of a ring A . If X is a ring and P is an X -module, an element u of P is called X -unimodular if $f(u)$ is a unit of X for some $f \in \text{Hom}_X(P, X)$. If P is a B -module, we regard P also as an A -module in the usual way.

PROPOSITION 2.1 [3]. *Suppose X is a ring and P is an X -module.*

(a) *If $u \in P$ is X -unimodular, then $u \notin mP$ for all $m \in \max X$.*

(b) *If P is a projective X -module and $u \notin mP$ for all $m \in \max X$, then u is X -unimodular.*

COROLLARY 2.2. *Suppose (B, A) has Property E_M and $aB \neq B$ for all proper ideals a of A . Suppose also that P is a B -module which is projective as an A -module. If $u \in P$ is B -unimodular, then u is A -unimodular.*

COROLLARY 2.3. *Suppose (B, A) has Property E_M and suppose P is a projective B -module. If $u \in P$ is A -unimodular, then u is also B -unimodular.*

The projective modulus of a ring X , denoted $\text{pm } X$, is the least nonnegative integer k such that every projective X -module is the direct sum of a free module and a module of rank $\leq k$.

COROLLARY 2.4. *Suppose B is a projective A -module for which $0 < f - \text{rank}_A B = d < \infty$ [see [1] for definition], $\max B$ is noetherian, and (B, A) has Property E_M . Then $\text{pm } B \leq \text{pm } A/d$. If B is also an integral extension of A , then $\text{pm } B \leq \dim \max B/d$.*

The proof is in [3].

The ring extension B in Corollary 2.4 for which $f - \text{rank}_A B = d > 1$ satisfies the inequality that $\dim \max B > \text{pm } B$. For such rings Corollary 2.4 is an improvement over Serre's theorem [1] that $\dim \max B \geq \text{pm } B$.

3. Integral extensions. Throughout this section we assume B is an integral extension of A . Let $\{P_i\}$ be a family of prime ideals of A and $T = A - \bigcup P_i$. The family $\{P_i\}$ is said to be of *maximal type* [6] if every maximal ideal of $T^{-1}A$

is of the form $T^{-1}P_i$ for some i , or equivalently if any ideal of A maximal among the ideals not intersecting T is some P_i . If there are no inclusion relations among the P_i 's then $\max T^{-1}A = \{T^{-1}P_i\}$ [6].

THEOREM 3.1. *Suppose $\{P_i\}$ is a family of prime ideals of A of maximal type such that there are no inclusion relations among the P_i 's. Let $\{Q_j\}$ be the family of all prime ideals Q of B such that $Q \cap A = P_i$ for some i . Let $T = A - \bigcup P_i$ and $S = B - \bigcup Q_j$. Then:*

- (a) $T^{-1}B = S^{-1}B$;
- (b) $\{Q_j\}$ is a family of prime ideals of B of maximal type such that there are no inclusion relations among the Q_j 's.

PROOF. There are no inclusion relations among the Q_j 's by the Cohen-Seidenberg going-up lemma. The major part of the proof consists of showing that $\max T^{-1}B = \{Q_j(T^{-1}B)\}$. For each j , $Q_j \cap T = \emptyset$ and $Q_j(T^{-1}B)$ is a prime ideal of $T^{-1}B$. The maximal ideals of $T^{-1}B$ are those ideals of the form $M(T^{-1}B)$ where M is an ideal of B which is maximal with respect to the property that $M \cap T = \emptyset$. Since $T^{-1}B$ is an integral extension of $T^{-1}A$, for such M 's $(M \cap A)T^{-1}A = M(T^{-1}B) \cap T^{-1}A$ is a maximal ideal of $T^{-1}A$. It follows that $M \cap A = P_i$ for some i . Then $M = Q_j$ for some j . We have proved $\max T^{-1}B \subseteq \{Q_j(T^{-1}B)\}$. Since any $Q_j(T^{-1}B)$ is contained in some maximal ideal of $T^{-1}B$ of the type $Q_k(T^{-1}B)$, it follows that $Q_j \subseteq Q_k$; hence $Q_j = Q_k$. To prove Condition (a) let $s \in S$. For all j , $s \notin Q_j$; hence $s \notin Q_j(T^{-1}B)$. Therefore s is a unit of $T^{-1}B$. Condition (b) is a trivial consequence of Condition (a) and the preceding remarks.

COROLLARY 3.2. *Suppose $\{P_i\}$ is a family of prime ideals of A of maximal type such that $P_iB = Q_i$ is a prime ideal of B for each i . Let $S = B - \bigcup Q_i$ and $T = A - \bigcup P_i$. Then:*

- (a) $S^{-1}B = T^{-1}B$.
- (b) $(S^{-1}B, T^{-1}A)$ has Property E_M .
- (c) $\max S^{-1}B$ is homeomorphic to $\max T^{-1}A$.
- (d) If B is a projective A -module of f -rank d ($0 < d < \infty$) and $\max B$ is noetherian, then $\text{pm } S^{-1}B \leq \dim \max S^{-1}B/d$.

PROOF. It follows from the Cohen-Seidenberg going-up lemma that for each i , Q_i is the unique prime ideal of B which lies over P_i . The corollary now follows from Theorem 3.1, Proposition 1.5, and Corollary 2.4.

EXAMPLE. Suppose F is a field which is not algebraically closed and K is a finite extension field of F . $F[X]$ denotes the ring of polynomials $F[X_1, \dots, X_n]$ in n indeterminates; similarly for $K[X]$. Let I be a proper ideal of $F[X]$ and V be a nonempty set in the variety of I in $F^n = F \times \dots \times F$. Let $J = IK[X]$. Then [3] we have $J \cap F[X] = I$ and the map $A = F[X]/I \rightarrow K[X]/J = B$ is a monomorphism, which we regard as an inclusion. We consider A as a ring of functions from V into F and B as a ring of functions from V into K . For each $v \in V$, let $M_v = \{f \in A: f(v) = 0\}$ and $N_v = \{f \in B: f(v) = 0\}$. $\{M_v: v \in V\}$ is a family of prime ideals of A of maximal type [6]. $\{N_v: v \in V\}$ is a family of prime ideals of B . For each $v \in V$, $N_v = M_vB$ [3]. Let $T = A - \bigcup M_v = \{f \in A: f(v) \neq 0 \text{ for all } v \in V\}$ and $S = B - \bigcup N_v$.

$= \{f \in B: f(v) \neq 0 \text{ for all } v \in V\}$. Corollary 3.2 applies in this case.

One example [3], [5] that the upper bound in Corollary 3.2(d) is the best possible in general is obtained by taking F to be the field of real numbers, K to be the field of complex numbers, and I to be the ideal of $F[X]$ generated by $1 - X_1^2 - X_2^2 - X_3^2$. In this case $1 = \text{pm } S^{-1}B = \dim \max S^{-1}B/2$.

In the remainder of this paper we are concerned with partial converses to Theorem 3.1.

THEOREM 3.3. *Suppose $\{Q_i\}$ is a family of prime ideals of B of maximal type. For each i , let $P_i = Q_i \cap A$. Let $S = B - \bigcup Q_i$ and $T = A - \bigcup P_i$. Suppose $T^{-1}B = S^{-1}B$. Then $\{P_i\}$ is a family of prime ideals of A of maximal type.*

PROOF. It is sufficient to show that $\max T^{-1}A \subseteq \{P_i(T^{-1}A)\}$. Suppose m is a maximal ideal of $T^{-1}A$. Since $T^{-1}B$ is an integral extension of $T^{-1}A$, there is a maximal ideal $Q_i(T^{-1}B) = Q_i(S^{-1}B)$ of $S^{-1}B$ which lies over m ; that is, $m = (Q_i \cap A)T^{-1}A = P_i(T^{-1}A)$.

THEOREM 3.4. *Suppose B , when regarded as an A -module, is finitely-generated and projective. Suppose $\{Q_i\}$ is a family of prime ideals of B of maximal type such that there are no inclusions among the Q_i 's and such that for each i , Q_i is the unique prime ideal of B which lies over the prime ideal $P_i = Q_i \cap A$ of A . Let $S = B - \bigcup Q_i$ and $T = A - \bigcup P_i$. Then:*

(a) $S^{-1}B = T^{-1}B$.

(b) $\{P_i\}$ is a family of prime ideals of A of maximal type such that there are no inclusion relations among the P_i 's.

In the proof we need to recall some properties of the determinant, denoted $\det f$, of an endomorphism f of a finitely-generated projective A -module M [2]. An A -module N is chosen so that $M \oplus N = F$ is a finitely-generated free A -module. Let g be the extension of f to the endomorphism of F which is the identity mapping on N . Then $\det f$ is defined to be $\det g$. f is an automorphism if and only if $\det f$ is a unit of A . If T is a multiplicatively closed set of A such that $0 \notin T$ and $1 \in T$, then the endomorphism $T^{-1}f$ of $T^{-1}P$ has the same determinant as f .

PROOF. Let $s \in S$. Let $f: B \rightarrow B$ be the A -endomorphism defined by $f(b) = sb$ for each $b \in B$. It is sufficient to prove that $T^{-1}f$ is an automorphism; for if g is the inverse of $T^{-1}f$, then $1 = T^{-1}f(g(1)) = sg(1)$ and then s is a unit of $T^{-1}B$. Since $\det T^{-1}f = \det f \in A$, it suffices to prove that $\det f \notin P_i$ for all i . Fix i and let $P = P_i$ and $Q = Q_i$. It follows from the hypothesis that B_P has a unique maximal ideal QB_P ; hence $B_P = B_Q$. Since s is a unit in B_Q , $(B - Q)^{-1}f$ is an A_P -automorphism of B_Q and $\det f = \det (B - Q)^{-1}f$ is a unit in A_P . Hence $\det f \notin P$. Condition (b) follows from Condition (a), Theorem 3.3 and the Cohen-Seidenberg going-up lemma.

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