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A NOTE ON THE SPACES L_p FOR $0 < p \leqslant 1$

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ABSTRACT. It is shown that there is no Hausdorff vector topology ρ on the space L_p (where $0 < p \leqslant 1$) such that the unit ball of L_p is relatively compact for the topology ρ.

It is well known that the space $L_1(0, 1)$ is not a dual Banach space; this follows from the Krein-Milman theorem. It is not even isomorphic to a dual space, by a result due to Gelfand [2] (see Bessaga and Pełczyński [1] and Namioka [6]). An equivalent statement is that there is no Hausdorff locally convex vector topology ρ on L_1 such that the unit ball of L_1 is relatively compact for ρ.

In this note we establish a conjecture due to J. H. Shapiro that for $0 < p \leqslant 1$ there is no Hausdorff vector topology on the space $L_p(0, 1)$ such that the unit ball is relatively compact. For the case $p = 1$, this extends the previous result as we no longer restrict the topology ρ to be locally convex. Note that for the space l_p, $0 < p \leqslant 1$, the topology of coordinatewise convergence makes the unit ball compact. Also the topology of uniform convergence on compact subsets of Δ, the open unit disc, makes the unit ball of H_p compact for $0 < p \leqslant 1$ (cf. [4]).

We shall suppose throughout that all vector spaces are real, although the extension to the complex case presents no problems. The norm on L_p is defined by

$$\|f\|_p = \int_0^1 |f(t)|^p \, dt$$

for $0 < p \leqslant 1$. We shall also need the space $L_\infty(0, 1)$ of essentially bounded functions with the norm

$$\|f\|_\infty = \operatorname{ess\,sup} |f(t)|.$$

We first gather together some general results.

PROPOSITION. *Let X be a separable complete p-normed space with unit ball U. Suppose that there exists on X a Hausdorff vector topology such that U is relatively compact. Then*

(i) *there is a metrizable vector topology γ on X such that U is γ-relatively compact*;

(ii) *if V is the γ-closure of U, then V is the unit ball of an equivalent p-norm* (*i.e. V is bounded*);

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(iii) *let* j: $(U, \gamma) \to (U, \tau)$ *denote the identity map and* τ *the norm topology on* U. *Then the points of continuity of* j *are dense in* (U, γ).

PROOF. (i) If δ is any Hausdorff vector topology such that U is relatively compact for δ, then there exists a metrizable vector topology $\gamma \leqslant \delta$ by an obvious modification of a result of Labuda [5]. Clearly U is also γ-relatively compact.

(ii) Clearly V is p-convex and the unit ball of a p-norm $\|\cdot\|^*$ on X, since V is compact in a vector topology on X. Then $(X, \|\cdot\|^*)$ is complete (use the γ-compactness of V) and by the Closed Graph Theorem V is bounded.

(iii) Consider j: $(V, \gamma) \to (V, \tau)$. By (ii) the norm topology has a base of γ-closed neighbourhoods of zero (i.e. is γ-polar) and hence by an obvious modification of Proposition 1.2 of Namioka [6] the points of continuity of j are dense in V. If $x \in V$ is such a point of continuity, then there is a sequence $x_n \to x(\gamma)$ with $x_n \in U$. Then $x_n \to x$ in norm and so $x \in C$.

Next we note two easy (and well-known) lemmas.

LEMMA 1. *Suppose* $0 < p \leqslant 1$ *and* $\phi \in L_p(0,1)$ *with* $\phi \neq 0$. *Define* T: $L_\infty(0,1) \to L_p(0,1)$ *by* $Th = h \cdot \phi$. *Then* T *is not compact.*

PROOF. We may suppose the existence of a measurable set E of positive measure such that $|\phi(t)| \geqslant \alpha > 0$ for $t \in E$. Then we may define measurable sets E_{mn}, $n = 1, 2, \ldots, 2^m$, $m = 1, 2, 3, \ldots$, such that:

(i) For each m, the sets E_{mn}, $n = 1, 2, \ldots, 2^m$, are disjoint and satisfy $\bigcup_n E_{mn} = E$;

(ii) $\lambda(E_{mn}) = 2^{-m}\lambda(E)$ where λ denotes Lebesgue measure;

(iii) $E_{mn} = E_{m+1,2n-1} \cup E_{m+1,2n}$.

Then let

$$h_m = \sum_{n=1}^{2^m} (-1)^n \chi(E_{mn}).$$

If $m \neq k$, $h_m(t) - h_k(t) \neq 0$ precisely on a subset of E of measure $\frac{1}{2}\lambda(E)$, where $h_m(t) - h_k(t) = 2$. Thus

$$\|Th_m - Th_k\|_p \geqslant 2^{p-1}\lambda(E)\alpha^p, \qquad m \neq k,$$

so that T is not compact.

LEMMA 2. *If* $-1 \leqslant x < \infty$ *and* $0 < p \leqslant 1$ *then* $(1+x)^p \leqslant 1 + px$.

PROOF. If $\psi(x) = 1 + px - (1+x)^p$, then $\psi'(x) = p - p(1+x)^{p-1}$. Thus $\psi'(x) \geqslant 0$ when $x \geqslant 0$ and $\psi'(x) \leqslant 0$ when $x \leqslant 0$. Hence $\psi(x) \geqslant \psi(0) = 0$.

THEOREM 1. *There is no Hausdorff vector topology on* L_p *for* $0 < p \leqslant 1$ *such that the unit ball of* L_p *is relatively compact.*

PROOF. Let U be the unit ball of L_p. If the theorem is false we may suppose the existence of a metrizable topology γ on L_p such that U is γ-relatively compact (by the Proposition). Again, using the Proposition, there exists $\phi \in U$ with $\phi \neq 0$ such that the identity map j: $(U, \gamma) \to (U, \tau)$ has a point of continuity at ϕ (τ denotes the norm topology).

Let G be the subspace of $L_\infty(0, 1)$ of all $h \in L_\infty$ such that $\int_0^1 h(t)|\phi(t)|^p\,dt = 0$. G is a closed subspace of codimension one, since $\int_0^1 |\phi(t)|^p\,dt < \infty$. Thus the map T: $G \to L_p$, defined by $Th = \phi \cdot h$, is not compact by Lemma 1. Let $h_n \in G$ be any sequence such that $\|h_n\|_\infty \leqslant 1$ and $\|Th_n - Th_m\|_p \geqslant \alpha > 0$, whenever $n \neq m$

By selection of subsequence, we may suppose that $\{Th_n\}$ is γ-convergent (U is relatively γ-compact). Hence $T(h_n - h_{n+1}) \to O(\gamma)$. Now let

$$g_n = \phi + \tfrac{1}{2}(Th_n - Th_{n+1}).$$

Then

$$\begin{aligned}|g_n(t)| &= |\phi(t)|\,|1 + \tfrac{1}{2}(h_n(t) - h_{n+1}(t))| \\ &= |\phi(t)|[1 + \tfrac{1}{2}(h_n(t) - h_{n+1}(t))].\end{aligned}$$

Hence

$$\begin{aligned}\|g_n\|_p &= \int_0^1 |\phi(t)|^p(1 + \tfrac{1}{2}(h_n(t) - h_{n+1}(t)))^p\,dt \\ &\leqslant \int_0^1 |\phi(t)|^p(1 + (p/2)(h_n(t) - h_{n+1}(t)))\,dt \\ &\leqslant 1 + (p/2)\int_0^1 |\phi(t)|^p(h_n(t) - h_{n+1}(t))\,dt = 1.\end{aligned}$$

Thus $g_n \in U$ and $g_n \to \phi$ in γ. Hence $\|g_n - \phi\|_p \to 0$, i.e. $\|Th_n - Th_{n+1}\| \to 0$, contradicting our choice of h_n. This proves the theorem.

Remark 1. In [7], Turpin notes that there are no known examples of nonzero compact operators T: $L_p \to E$, where $0 < p < 1$ and E is any topological vector space. It is thus unknown whether there exists a Hausdorff vector topology on L_p for which U is precompact. (See note added in proof.)

Remark 2. We are indebted to Professor Shapiro for the observation that the quotient space $H_p|qH_p$ constructed in [4] is an example of a locally bounded space with trivial dual, but such that there is a Hausdorff vector topology making the unit ball compact. This latter topology is the quotient β-topology. In particular, note that $H_p|qH_p$ is not isomorphic to L_p.

Remark 3. A number of Banach spaces, which are known not to be dual spaces, can also be shown not to admit any Hausdorff vector topology making the unit ball compact by the following:

THEOREM 2. *Let X be a separable F-space containing an isomorphic copy of c_0. Then there does not exist a Hausdorff vector topology γ on X such that every bounded set is relatively γ-compact.*

PROOF. Suppose γ exists. It is easy to show that if (e_n) is the unit vector basis of $c_0 \subset X$, then $\sum e_n$ converges subseries in (X, γ). By Theorem 3 of [3], $\sum e_n$ converges in X, which is a contradiction.

COROLLARY. *Theorem* 1 *also applies to $C(X)$ for X compact metric, and $K(H)$, the space of compact operators on a separable Hilbert space.*

ADDED IN PROOF. The author has now shown that there are no compact operators with domain L_p $(0 < p < 1)$ (cf. Remark 1).

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