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ON THE MEAN ERGODIC THEOREM OF SINE

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ABSTRACT. Robert Sine has shown that $(1/n)(I + T + \cdots + T^{n-1})$, the ergodic averages, converge in the strong operator topology iff the invariant vectors of T separate the invariant vectors of the adjoint operator T^*, T being any Banach space contraction. We prove a generalization in which (spectral radius of T) $\leqq 1$ replaces $\|T\| \leqq 1$, and any bounded averaging sequence converging uniformly to invariance replaces the ergodic averages; it is necessary to assume that such sequences exist.

1. **Introduction.** Let $T\colon \mathfrak{X} \to \mathfrak{X}$ be a bounded linear operator of spectral radius $r(T) \leqq 1$ on real or complex Banach space $\mathfrak{X}$, let $\mathfrak{M} = \{x \in \mathfrak{X}\colon Tx = x\}$ be the invariant vectors of T, and let $\mathfrak{N} = \{\xi \in \mathfrak{X}^*\colon T^*\xi = \xi\}$ be the invariant vectors of the adjoint operator $T^*\colon \mathfrak{X}^* \to \mathfrak{X}^*$. Robert Sine has shown [4] that in the case $\|T\| \leqq 1$, the condition $\langle \mathfrak{M}$ separates $\mathfrak{N}\rangle$ is necessary and sufficient for strong convergence of the ergodic averages $(1/n)(I + T + \cdots + T^{n-1})x$, $x \in \mathfrak{X}$. We show here that $\langle \mathfrak{M}$ separates $\mathfrak{N}\rangle$ is necessary and sufficient for strong convergence of any bounded regular invariant summability method, and that the limiting operator, a projection onto $\mathfrak{M}$, is independent of the method when it exists.

2. **Mean ergodic theorem.** Let $\mathcal{Q}_1$ be the set of power series $p(z) = \sum_{i=0}^{\infty} p_i z^i$ which have radius of convergence greater than 1 and are such that $\sum_{i=0}^{\infty} p_i = 1$. With $\mathfrak{B}(\mathfrak{X})$ the bounded linear operators on $\mathfrak{X}$, each $p \in \mathcal{Q}_1$ determines $p(T) = \sum_{i=0}^{\infty} p_i T^i \in \mathfrak{B}(\mathfrak{X})$, the series converging in the uniform operator topology. We put $\mathcal{P}_1 = \{p(T)\colon p \in \mathcal{Q}_1\}$, noting that if $P \in \mathcal{P}_1$ then $Px = x$, $x \in \mathfrak{M}$.

We introduce further $\mathcal{P}_1(M) = \{P \in \mathcal{P}_1\colon \|P\| \leqq M\}$ for $1 \leqq M < \infty$, and also $\mathcal{P}_1(M,\varepsilon) = \{P \in \mathcal{P}_1(M)\colon \|(I - T)P\| \leqq \varepsilon\}$, $1 \leqq M < \infty$, $\varepsilon > 0$. Our basic assumption will be:

(UI) There exists $1 \leqq M_0 < \infty$ such that
$\mathcal{P}_1(M_0, \varepsilon) \neq \varnothing$ for each $\varepsilon > 0$.

In the terminology of Day [1], this is the assertion that $\mathcal{P}_1$ contains bounded sequences $\{P_n\}$ converging uniformly to invariance, i.e., $\lim_n \|P_n - TP_n\| = 0$. In §3 we give various sufficient conditions for (UI).

Let $\overline{\mathcal{P}}_1$ be the closure of $\mathcal{P}_1 \subset \mathfrak{B}(\mathfrak{X})$ in the strong operator topology (SOT), and let $\overline{\mathcal{P}}_1(M,\varepsilon)$ be the SOT closure of $\mathcal{P}_1(M,\varepsilon)$.

THEOREM 1. *If* (UI) *holds then* (i) *is equivalent to* (ii):

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(1) (i) $\mathfrak{M}$ *separates* $\mathfrak{N}$;
(ii) *There exists a projection* $Q \in \overline{\mathcal{P}}_1$ *onto* $\mathfrak{M}$, *necessarily unique, and* $\bigcap_{\varepsilon>0}\overline{\mathcal{P}}_1(M,\varepsilon) = \{Q\}$ *for any* $M \geqq M_0$.

PROOF. We show first that $\overline{\mathcal{P}}_1$ contains at most one projection Q onto $\mathfrak{M}$. Suppose $\lim_\nu P_\nu = Q$ in SOT with $\{P_\nu\} \subset \mathcal{P}_1$ a generalized sequence and Q a projection onto $\mathfrak{M}$. Such a projection commutes with T, from

$$Q = TQ = T\lim_\nu P_\nu = \lim_\nu TP_\nu = \lim_\nu P_\nu T = QT,$$

the limits being SOT. If $\lim_\nu P_\nu' = Q'$ and $\lim_\nu P_\nu'' = Q''$, then $Q' = Q'P_\nu'' = Q'Q'' = Q''$, whence Q is unique in $\overline{\mathcal{P}}_1$ if it exists.

(ii $\Rightarrow$ i) Let us prove that if projection Q exists then the adjoint projection $Q^*: \mathfrak{X}^* \to \mathfrak{X}^*$ is onto $\mathfrak{N}$. From $Q = TQ = QT$ follows $Q^* = Q^*T^* = T^*Q^*$, so that Q^* has range in $\mathfrak{N}$. If $Q = \lim_\nu P_\nu$ with $\{P_\nu\} \subset \mathcal{P}_1$ and if $x \in \mathfrak{X}$, $\xi \in \mathfrak{N}$, then

$$(x,\xi) = (x, P_\nu^*\xi) = (P_\nu x, \xi) = (Qx,\xi) = (x, Q^*\xi),$$

giving $Q^*\xi = \xi$, $\xi \in \mathfrak{N}$, whence Q^* is onto $\mathfrak{N}$.

Suppose $0 \neq \xi \in \mathfrak{N}$, and let $x \in \mathfrak{X}$ be such that $(x,\xi) \neq 0$. Then $(Qx,\xi) = (x,Q^*\xi) = (x,\xi) \neq 0$, so that $Qx \in \mathfrak{M}$ separates ξ and 0. That is, $\mathfrak{M}$ separates $\mathfrak{N}$.

(i $\Rightarrow$ ii) Let functional $\Phi(x)$, $x \in \mathfrak{X}$, be defined by

$$\Phi(x) = \lim_{\varepsilon\downarrow 0}\sup_{P',P''}\{\|P'x - P''x\|: P',P'' \in \mathcal{P}_1(M,\varepsilon)\}, \qquad x \in \mathfrak{X},$$

for any fixed $M \geqq M_0$. The properties

(a) $\Phi(x+y) \leqq \Phi(x) + \Phi(y)$, $x, y \in \mathfrak{X}$,
(b) $\Phi(cx) = |c|\Phi(x)$, $x \in \mathfrak{X}$, scalar c,
(c) $0 \leqq \Phi(x) \leqq 2M\|x\|$, $x \in \mathfrak{X}$,
(d) $\Phi(x) = 0$, $x \in \mathfrak{M}$,
(e) $\Phi(x - Tx) = 0$, $x \in \mathfrak{X}$,

are straightforward or obvious; e.g. for (e) we use

$$\begin{aligned}\|P'(x-Tx) - P''(x-Tx)\| &\leqq [\|(I-T)P'\| + \|(I-T)P''\|]\|x\| \\ &\leqq 2\varepsilon\|x\|, \qquad x \in \mathfrak{X} \text{ and } P',\, P'' \in \mathcal{P}_1(M,\varepsilon).\end{aligned}$$

Let Φ^0 denote the set of all $\xi \in \mathfrak{X}^*$ satisfying $(x,\xi) \leqq \Phi(x)$, $x \in \mathfrak{X}$; by the Hahn-Banach theorem, $\Phi(x) = \max_\xi\{(x,\xi): \xi \in \Phi^0\}$, $x \in \mathfrak{X}$. Properties (d), (e) of Φ yield properties: (d*) $\xi \in \mathfrak{M}^\perp$; (e*) $\xi \in \mathfrak{N}$, for the $\xi \in \Phi^0$, where $\mathfrak{M}^\perp \subset \mathfrak{X}^*$ is the annihilator of $\mathfrak{M}$. Now, condition (1)(i) is the assertion $\mathfrak{M}^\perp \cap \mathfrak{N} = 0$; from $\Phi^0 \subset \mathfrak{M}^\perp \cap \mathfrak{N}$ just shown follows then $\Phi^0 = 0$ and hence $\Phi = 0$.

Let $\{\eta_n\}$ be any sequence of positive numbers such that $\lim_n \eta_n = 0$, let P_n for each n be an arbitrarily chosen member of $\mathcal{P}_1(M,\eta_n)$, and put $\varepsilon_n = \max_{r\geqq n}\eta_r$. Any such sequence $\{P_n\}$ is SOT Cauchy when $\Phi = 0$, from

$$\begin{aligned}\lim_{n\to\infty}\sup_{n\leqq r<s}\|P_rx - P_sx\| &\leqq \lim_{n\to\infty}\sup_{P',P''}\{\|P'x - P''x\|: P',P'' \in \mathcal{P}_1(M,\varepsilon_n)\}\\ &= \lim_{\varepsilon\downarrow 0}\sup_{P',P''}\{\|P'x - P''x\|: P',P'' \in \mathcal{P}_1(M,\varepsilon)\}\\ &= \Phi(x) = 0, \qquad x \in \mathfrak{X};\end{aligned}$$

we have used the nested property of the $\{\mathcal{P}_1(M,\varepsilon): \varepsilon > 0\}$. The SOT limit $Q = \lim_n P_n$ is the projection sought. □

We remark that when $\mathfrak{M} = 0$ the result takes the form: given (UI), $0 \in \overline{\mathcal{P}}_1$ iff $\mathfrak{N} = 0$.

3. **Condition** (UI). Consider the familiar ergodic averages $A_n = (1/n)\sum_{i=0}^{n-1} T^i$, $n \geqq 1$, for which $(I - T)A_n = (I - T^n)/n$. If these satisfy $\|A_n\| \leqq M_0 < \infty$, $n \geqq 1$, and if $\liminf_n \|T^n\|/n = 0$, then (UI) holds, clearly. A fortiori, (UI) holds when $\|T^n\| \leqq M_0 < \infty$, $n \geqq 1$; this case is covered by the arguments of [4], although Sine assumes $\|T\| \leqq 1$.

The resolvent of T is given by $R_\lambda = \sum_{i=0}^{\infty} T^i/\lambda^{i+1}$ when $|\lambda| > 1 \geqq r(T)$, the series converging in the uniform operator topology. If we introduce $P_\lambda = (\lambda - 1)R_\lambda$ for $|\lambda| > 1$, then $P_\lambda \in \mathcal{P}_1$, and it is easily verified that $(I - T)P_\lambda = (\lambda - 1)(I - P_\lambda)$. Thus if the $\{P_\lambda\}$ satisfy

$$\liminf_{\lambda\to 1;|\lambda|>1} \|P_\lambda\| < M_0 < \infty, \tag{2}$$

then (UI) holds for such M_0.

In the other direction, suppose T has index $1 < \mu < \infty$ at $\lambda = 1$. (The index of T at $\lambda = 1$ is the least integer $\mu \geqq 0$ with the property: all vectors $x \in \mathfrak{X}$ satisfying $(I - T)^{\mu+1}x = 0$ satisfy also $(I - T)^\mu x = 0$ [2, p. 556].)

THEOREM 2. *If T has index $1 < \mu < \infty$ at $\lambda = 1$ then* (UI) *cannot hold and $\overline{\mathcal{P}}_1$ contains no projection onto $\mathfrak{M}$.*

PROOF. If a generalized sequence $\{P_\nu\} \subset \mathcal{P}_1$ is SOT convergent to a projection $Q \in \overline{\mathcal{P}}_1$ onto $\mathfrak{M}$, then, necessarily,

$$\lim_\nu \|(I - T)P_\nu x\| = \|(I - T)Qx\| = 0, \qquad x \in \mathfrak{X}.$$

If T has index $1 < \mu < \infty$ at $\lambda = 1$, then unit vectors $x_1, x_2 \in \mathfrak{X}$ exist such that $Tx_1 = x_1$, $Tx_2 = x_2 + cx_1$ for some $c > 0$. From

$$(I - T)T^i x_2 = T^i(I - T)x_2 = T_i(-cx_1) = -cx_1, \qquad i \geqq 0,$$

follows $(I - T)Px_2 = -cx_1$, $P \in \mathcal{P}_1$. We have then $\|(I - T)Px_2\| = c > 0$, $P \in \mathcal{P}_1$, showing that no Q exists, and $\|(I - T)P\| \geqq c > 0$, $P \in \mathcal{P}_1$, showing that (UI) fails. □

For the same x_1, x_2 an easy calculation gives $P_\lambda x_2 = x_2 + cx_1/(\lambda - 1)$, whence

$$\begin{aligned}\|P_\lambda\| &\geqq |c/|\lambda - 1| - 1|, &&|\lambda| > 1,\\ &= O(1/|\lambda - 1|), &&|\lambda| > 1 \text{ and } \lambda \to 1,\end{aligned}$$

when T has index $1 < \mu < \infty$ at $\lambda = 1$. Thus if some condition on $\|P_\lambda\|$, $|\lambda| > 1$, $\lambda \to 1$, is necessary and sufficient for (UI) then it lies between (2) and

$$\|P_\lambda\| = o(1/|\lambda - 1|), \qquad |\lambda| > 1, \lambda \to 1.$$

Apart from changes in variable, the following result is the (C, α) generalization of Theorem 1′ of [3]. For $\alpha \neq -1, -2, \ldots$ the (C, α) averages $A_n^{(\alpha)}(T)$ of $\{T^n\}$ have as generating function

$$\left(1 - \frac{1}{\lambda}\right)^{-\alpha} R_\lambda(T) = \sum_{n=0}^{\infty} \frac{\alpha + n!}{\alpha!\, n!} \frac{A_n^{(\alpha)}(T)}{\lambda^{n+1}}; \qquad |\lambda| > 1,$$

so that

$$A_n^{(\alpha)}(T) = \frac{\alpha!\, n!}{\alpha + n!} \frac{1}{2\pi i} \int \lambda^n \left(1 - \frac{1}{\lambda}\right)^{-\alpha R_\lambda} (T)\, d\lambda, \qquad n \geqq 0,$$

the contour being a large circle around the origin, say.

For λ in the resolvent set of T let $T_\lambda \in \mathfrak{B}(\mathfrak{X})$ be defined by

$$T_\lambda = P_\lambda T = (\lambda - 1)(\lambda I - T)^{-1} T = \lambda P_\lambda - (\lambda - 1) I.$$

The spectrum of T_λ is $\sigma(T_\lambda) = \{(\lambda - 1)\lambda'/(\lambda - \lambda') \colon \lambda' \in \sigma(T)\}$, and we find $1 - \lambda \notin \sigma(T_\lambda)$ provided $\lambda \neq 0, 1$. Thus the inversion formula

$$T = \lambda T_\lambda [(\lambda - 1) I + T_\lambda]^{-1}$$

is valid for $\lambda \neq 0, 1$ in the resolvent set of T; moreover, $r(T_\lambda/(1 - \lambda)) < 1$ if $|\lambda| > 2$. Note that the relations between T_λ and T admit the involution $T \leftrightarrow T_\lambda$, $\lambda \leftrightarrow 1 - \lambda$.

THEOREM 3. *For any fixed* $\alpha \geqq 0$, $1 \leqq M < \infty$, *the conditions*

$$\text{(3)} \qquad \begin{aligned} &\text{(i) } \|A_n^{(\alpha)}(T)\| \leqq M, && n \geqq 0, \\ &\text{(ii) } \|A_n^{(\alpha)}(T_\lambda)\| \leqq M \left| \frac{\lambda - 1}{|\lambda| - 1} \right|^{\alpha + n}, && n \geqq 0, |\lambda| > 1, \end{aligned}$$

are equivalent. When they are satisfied, (UI) *holds and*

$$\text{(4)} \qquad \|P_\lambda(T)\| \leqq M |(\lambda - 1)/(|\lambda| - 1)|^{\alpha + 1}, \qquad |\lambda| > 1.$$

PROOF. In

$$A_n^{(\alpha)}(T_\lambda) = \frac{\alpha!\, n!}{\alpha + n!} \frac{1}{2\pi i} \int \mu^n \left(1 - \frac{1}{\mu}\right)^{-\alpha} R_\mu(T_\lambda)\, d\mu, \qquad n \geqq 0,$$

we have

$$R_\mu(T_\lambda) = \frac{1}{\mu I - T_\lambda} = \frac{1}{\lambda + \mu - 1} \left\{ I + \frac{\lambda(\lambda - 1)}{\lambda \mu I - (\lambda + \mu - 1) T} \right\};$$

the change of variable $\xi = \lambda\mu/(\lambda + \mu - 1)$ gives

$$A_n^{(\alpha)}(T_\lambda)$$
$$= \frac{\alpha!\,n!}{\alpha+n!}\left(1-\frac{1}{\lambda}\right)^{\alpha+n}\frac{1}{2\pi i}\int\left[\frac{\xi}{1-(\xi/\lambda)}\right]^n\left(1-\frac{1}{\xi}\right)^{-\alpha}\left[\frac{I}{\xi-\lambda}+R_\xi(T)\right]d\xi$$
$$= \left(1-\frac{1}{\lambda}\right)^{\alpha+n}\sum_{j=0}^{\infty}\frac{\alpha+n-1+j!}{\alpha+n-1!\,j!\,\lambda^j}\left\{\frac{\alpha jI}{(\alpha+n)(j+n)} + \left[1-\frac{\alpha j}{(\alpha+n)(j+n)}\right]A_{n+j}^{(\alpha)}(T)\right\},$$
$$n \geqq 0,\ |\lambda| > 1,$$

with $1 < |\xi| < |\lambda|$ on the contour. If $\alpha \geqq 0$ then $0 \leqq \alpha/(\alpha+n)\cdot j/(j+n) \leqq 1$ in the last expression, so if (3)(i) holds, then

$$\|A_n^{(\alpha)}(T_\lambda)\| \leqq \left|1-\frac{1}{\lambda}\right|^{\alpha+n}\left(1-\frac{1}{|\lambda|}\right)^{-\alpha-n}M = \left|\frac{\lambda-1}{|\lambda|-1}\right|^{\alpha+n}M, \qquad |\lambda| > 1,$$

which is (3)(ii).

In the same way, using the bound (3)(ii) in the inversion formula

$$A_n^{(\alpha)}(T) = \left(1-\frac{1}{\lambda}\right)^{-\alpha-n}\cdot\sum_{j=0}^{\infty}\frac{\alpha+n-1+j!}{\alpha+n-1!\,j!\,(1-\lambda)^j}\left\{\frac{\alpha jI}{(\alpha+n)(j+n)} + \left[1-\frac{\alpha j}{(\alpha+n)(j+n)}\right]A_{n+j}^{(\alpha)}(T_\lambda)\right\},$$
$$n \geqq 0,\ |\lambda| > 2,$$

gives

$$\|A_n^{(\alpha)}(T)\| \leqq [|\lambda|/(|\lambda|-2)]^{\alpha+n}M, \qquad n \geqq 0,\ |\lambda| > 2;$$

we let $|\lambda| \to \infty$ to obtain (3)(i).

If conditions (3) are satisfied then

$$\|R_\lambda(T)\| \leqq \left[\frac{|\lambda-1|}{|\lambda|}\right]^{\alpha}\left[1-\frac{1}{|\lambda|}\right]^{-\alpha-1}\frac{M}{|\lambda|} = \frac{|\lambda-1|^{\alpha}M}{(|\lambda|-1)^{\alpha+1}}, \qquad |\lambda| > 1,$$

which is (4). For a (UI) sequence we may take $\{P_{\lambda_n}\}$ for some $\{\lambda_n \downarrow 1\}$, or $\{A_n^{(\alpha+1)}(T)\}$, since

$$A_n^{(\alpha+1)}(T) = \sum_{j=0}^{n}\frac{\alpha+j!}{\alpha!\,j!}A_j^{(\alpha)}(T)\Big/\sum_{j=0}^{n}\frac{\alpha+j!}{\alpha!\,j!},$$
$$(I-T)A_n^{(\alpha+1)}(T) = (\alpha+1)/(n+1)[I-A_{n+1}^{(\alpha)}(T)], \qquad n \geqq 0. \quad \square$$

Note the slight strenghtening of the result of the first paragraph: if $\|A_n^{(1)}(T)\| \leqq M < \infty$, $n \geqq 0$, then (UI) holds, with no condition on $\|T^n\|/n$. All that is involved in Theorem 3 is $\|T - T_\lambda\| = O(1/|\lambda|)$ at $|\lambda| \to \infty$, of course; the interesting part at $\lambda \to 1$ has no force.

4. **The adjoint projections.** Recall that a generalized sequence $\{\Gamma_\nu\} \subset \mathfrak{B}(\mathfrak{X}^*)$ is W* OT convergent to $\Gamma \in \mathfrak{B}(\mathfrak{X}^*)$ iff $\lim_\nu(x, \Gamma_\nu \xi) = (x, \Gamma\xi)$ for each $x \in \mathfrak{X}$, $\xi \in \mathfrak{X}^*$; and further, that bounded W* OT closed sets are W* OT compact. Let $\mathcal{P}_1^*(M,\varepsilon) \subset \mathfrak{B}(\mathfrak{X}^*)$ be the set of adjoints of members of $\mathcal{P}_1(M,\varepsilon)$, and let $\overline{\mathcal{P}}_1^*(M,\varepsilon)$ denote the W* OT closure of $\mathcal{P}_1^*(M,\varepsilon)$. If (UI) holds and $M \geqq M_0$, then $\{\overline{\mathcal{P}}_1^*(M,\varepsilon): \varepsilon > 0\}$ is a nested family of nonempty convex W* OT compact sets; the intersection $\mathcal{S}(M) = \bigcap_{\varepsilon>0}\overline{\mathcal{P}}_1^*(M,\varepsilon)$, necessarily nonempty, consists of projections onto $\mathfrak{N}$ (with $\mathcal{S}(M) = \{0\}$ if $\mathfrak{N} = 0$). The members of $\mathcal{S}(M)$ commute with T^* and satisfy $\Gamma'\Gamma'' = \Gamma''$, $\Gamma', \Gamma'' \in \mathcal{S}(M)$, clearly. If the projection $Q \in \overline{\mathcal{P}}_1$ onto $\mathfrak{M}$ exists, then it satisfies $Q^*\Gamma = Q^*$, $\Gamma \in \mathcal{S}(M)$. For, suppose $\lim_\nu P_\nu^* = \Gamma \in \mathcal{S}(M)$ in W* OT; then

$$\begin{aligned}(x, Q^*\Gamma\xi) &= (Qx, \Gamma\xi) = \lim_\nu (Qx, P_\nu^*\xi) = \lim_\nu (P_\nu Qx, \xi)\\ &= (Qx, \xi) = (x, Q^*\xi), \qquad x \in \mathfrak{X}, \xi \in \mathfrak{X}^*.\end{aligned}$$

This and $Q^*\Gamma = \Gamma$ give $\Gamma = Q^*$, which is to say, $\mathcal{S}(M)$ is the singleton $\mathcal{S}(M) = \{Q^*\}$ when Q exists. The author is not able to prove the following plausible converse: If $\mathcal{S}(M) = \{\Gamma\}$ is a singleton, then $\Gamma = Q^*$ for $Q \in \overline{\mathcal{P}}_1$ a projection onto $\mathfrak{M}$.

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