A CHARACTERIZATION OF $F^+ \cap N$

M. STOLL

ABSTRACT. In this note we give a characterization of $F^+ \cap N$, where N denotes the Nevanlinna class of functions of bounded characteristic and F^+ denotes the containing Fréchet space of N^+ . We show that a holomorphic function $f \in F^+ \cap N$ if and only if $f(z) = g(z)/S_\mu(z)$, where $g \in N^+$ and S_μ is a singular inner function with respect to a nonnegative continuous singular measure μ .

Let U denote the unit disc $\{|z| < 1\}$ and let T denote the boundary of U . As in [1], we denote by N the Nevanlinna class of functions of bounded characteristic on U and by N^+ the subspace of N consisting of those functions $f \in N$ for which

$$\lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})| dt = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(e^{it})| dt,$$

where $f(e^{it}) = \lim_{r \rightarrow 1^-} f(re^{it})$ a.e. on T . The space N^+ with the metric ρ given by

$$(1) \quad \rho(f, g) = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(re^{it}) - g(re^{it})|) dt$$

is an F -space [5]. The space F^+ , the containing Fréchet space of N^+ , consists of those functions f holomorphic in U satisfying

$$(2) \quad \limsup_{r \rightarrow 1^-} (1 - r) \log M(r, f) \leq 0$$

where $M(r, f) = \max_{|z|=r} |f(z)|$ [6].

Every function $f \in N$ can be factored uniquely as follows [1, p. 25]:

$$(3) \quad f(z) = B(z)(S_{\mu_1}(z)/S_{\mu_2}(z))F(z),$$

where $B(z)$ is the Blaschke product with respect to the zeros of $f(z)$, $F(z)$ is an outer function, and $S_{\mu_1}(z)$ and $S_{\mu_2}(z)$ are singular inner functions with respect to the nonnegative singular measures μ_1 and μ_2 and are given by

$$(4) \quad S_{\mu_j}(z) = \exp \left[- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu_j(t) \right], \quad j = 1, 2.$$

A function $f(z) \in N$ is in N^+ if and only if $S_{\mu_2}(z) \equiv 1$, i.e., $\mu_2 = 0$.

In [7], N. Yanagihara has constructed an example of a function $f \in F^+$

Received by the editors January 7, 1975 and, in revised form, April 21, 1975.

AMS (MOS) subject classifications (1970). Primary 30A76, 31A20.

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$\cap N$ but not in N^+ . In this note, using some results of [3], [4] we obtain the following characterization of $F^+ \cap N$.

THEOREM. $F^+ \cap N = \{f/S_\mu: f \in N^+, \mu \text{ a nonnegative continuous singular measure}\}$, where S_μ is given by (4).

PROOF. Recall that a measure μ on T is continuous if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in T$. As in [4], for each $\omega \in T$, we define λ_ω on N by

$$\lambda_\omega(f) = \limsup_{r \rightarrow 1^-} (1-r) \log^+ |f(r\omega)|.$$

By (3) every $f \in N$ can be written as $f(z) = g(z)/S_\mu(z)$, where $g \in N^+$, S_μ is a singular inner function given by (4), and the singular inner function in the factorization (3) of g and S_μ has no common divisor. By Theorem (2.2) of [4], if $f \in N$ and $\omega \in T$, $\lambda_\omega(f) = 2\mu(\{\omega\})$. Consequently, if $f \in F^+ \cap N$, by (2), $\lambda_\omega(f) = 0$ for all $\omega \in T$ and hence by the above, $\mu(\{\omega\}) = 0$ for all $\omega \in T$, i.e., μ is continuous.

Conversely, suppose $f = g/S_\mu$, where $g \in N^+$, and μ is a nonnegative continuous singular measure. Clearly $f \in N$, and since F^+ is closed under multiplication, it suffices to show that $1/S_\mu \in F^+$. By the remark on p. 164 of [3],

$$|S_\mu(re^{it})| > \exp[-C\omega(1-r)/(1-r)]$$

where $\omega(h)$ is the modulus of continuity of the distribution function of μ , and C is a positive constant. Consequently, since μ is continuous,

$$\limsup_{r \rightarrow 1^-} (1-r) \log M(r, 1/S_\mu) \leq 0.$$

Therefore by (2), $1/S_\mu \in F^+$, which proves the result.

REMARK. In [4], J. H. Shapiro and A. L. Shields have shown that the space N with the metric given by (1) is disconnected and they posed the problem of determining the component of the origin in N . This question has been answered by J. W. Roberts [2] where he has shown, using the above characterization of $F^+ \cap N$, that the component of the origin in N is $F^+ \cap N$.

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