THE STRUCTURE OF REGULATED FUNCTIONS

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ABSTRACT. It is shown that there is a nontrivial class of regulated functions each of which is representable as the sum of a continuous function and a uniformly convergent series of jump functions whose jumps are those of the given function. The set of regulated functions is the union of the classes of functions of bounded Φ -variation for convex Φ .

The regulated functions on a closed interval are those functions whose right and left limits exist at each point. Every regulated function is bounded, has a countable set of discontinuities, and is the limit of a uniformly convergent sequence of step functions. The regulated functions are of importance in the theory of stochastic processes and in the theory of everywhere convergence of Fourier series.

Functions of bounded variation are regulated, as are the functions of bounded Φ -variation [1] and the functions of bounded Λ -variation [3]. In §1 we shall show that each regulated function is of bounded Φ -variation for some Φ .

A function of bounded variation has a canonical representation as the sum of a continuous function of bounded variation and a sum of jump functions, the jumps being those of the given function of bounded variation. In §2 we shall investigate the possibility that a regulated function have a representation as the sum of a continuous function and a uniformly convergent series of jump functions whose jumps are those of the given function. We shall see that, although we can characterize a nontrivial class of functions for which this representation is possible, we can construct functions which have no such representation.

1. Let $\Phi(x)$ be a convex function on $[0, \infty)$ with $\Phi(0) = 0$, $\Phi(x) > 0$ for $x > 0$. A function f defined on an interval I is said to be of bounded Φ -variation ($\Phi - BV$) if the Φ -variation of f ,

$$V_{\Phi}(f) = \sup\{\sum \Phi(|f(I_n)|) : \{I_n\}\} < \infty,$$

where $\{I_n\}$ denotes a partition of I into intervals $[a_n, b_n]$ and $f(I_n) = f(b_n) - f(a_n)$. It is usual to consider only those Φ for which $\Phi(x)/x \rightarrow 0$ as $x \rightarrow 0$ in order that $\Phi - BV$ should contain BV properly.

It is clear that every function of $\Phi - BV$ is regulated. We shall suppose that at each point of discontinuity of a regulated function f ,

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$$\min\{f(x+), f(x-)\} \leq f(x) \leq \max\{f(x+), f(x-)\}.$$

We shall show that the class of regulated functions is the union of the classes $\Phi - BV$. An analogous theorem for functions of bounded Λ -variation has been obtained by S. J. Perlman [2].

THEOREM 1. *If f is regulated, then f is of bounded Φ -variation for some Φ with $\Phi(x)/x \rightarrow 0$ as $x \rightarrow 0$.*

PROOF. Let the domain of f be $[0, 1]$. We suppose, at first, that f is continuous. Let ω be the modulus of continuity of f and choose $k_n \searrow 0$, $n = 1, 2, \dots$, such that

$$\sum n k_n < \infty \quad \text{and} \quad k_n/\omega(1/n) \rightarrow 0.$$

Let Φ be any convex function on $[0, \infty)$ whose graph on $[0, \omega(1)]$ coincides with the lower boundary of the convex hull of $\{(\omega(1/n), k_n): n = 1, 2, \dots\}$. Then

$$\sum n \Phi(\omega(1/n)) < \infty \quad \text{and} \quad \Phi(x)/x \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

If $\{I_k\}$ is a partition of $[0, 1]$, then

$$\sum \Phi(|f(I_k)|) \leq \sum n \Phi(\omega(1/n)) < \infty,$$

implying that $f \in \Phi - BV$.

Supposing now that f is regulated on $[0, 1]$, let $\{a_n\}$ be the points of discontinuity of f . Let

$$y(x) = x + \sum_{n: a_n < x} 2^{-n}, \quad x \notin \{a_n\},$$

and

$$F(y(x)) = f(x), \quad x \notin \{a_n\}.$$

Let

$$t_n^- = \lim_{x \nearrow a_n} y(x), \quad t_n^+ = \lim_{x \searrow a_n} y(x).$$

On each $[t_n^-, t_n^+]$, define $F(t)$ to be the linear function whose graph is the segment connecting $(t_n^-, f(a_n -))$ and $(t_n^+, f(a_n +))$. Then $G(t) = F(t/2)$ is continuous on $[0, 1]$ and there is a Φ , as above, such that $G \in \Phi - BV$. For any partition $\{J_n\}$ of $[0, 1]$, there exists a collection of nonoverlapping intervals $J_n \subset [0, 1]$ such that

$$\sum \Phi(|f(J_n)|) = \sum \Phi(|G(J_n)|) \leq V_\Phi(G) < \infty,$$

and so $f \in \Phi - BV$.

2. Let f be a regulated function on $[a, b]$. We assume now that f is left continuous everywhere and continuous at b . Let the points of discontinuity of f be $\{a_n\}$, $n = 1, 2, \dots$, and the *saltus* of f at a_n is then $k_n = f(a_n +) - f(a_n)$. Then, although $\lim k_n = 0$, the series $\sum k_n$ need not converge. A *jump function* J_c^k is defined by

$$J_c^k(x) = 0, \quad x \leq c \quad \text{and} \quad J_c^k(x) = k, \quad x > c.$$

Our problem is to decide for which regulated f with saltus k_n at a_n , $n = 1, 2, \dots$, there is a rearrangement $\{a_{n_i}\}$ of $\{a_n\}$ and a continuous function g such that

$$(*) \quad f = g + \sum_{i=1}^{\infty} J_{a_{n_i}}^{k_{n_i}},$$

the series being uniformly convergent.

We define the *saltus function* f^* of f by

$$f^*(x) = f(x+) - f(x-).$$

Then f^* is zero except on a countable set. For each $\varepsilon > 0$, $\{x: |f^*(x)| > \varepsilon\}$ is finite. We shall first show that this is the essential property of the saltus function of a regulated function.

THEOREM 2. *If h is a function on $[a, b]$ such that $\{x: |h(x)| > 1/n\}$ is finite for $n = 1, 2, \dots$, and $h(a) = h(b) = 0$, then there is a regulated function f such that $f^* = h$.*

PROOF. Without loss of generality we may assume $\max |h(x)| \leq 1$. For a given positive integer n , let the finite set $\{x: 2^{-n} < |h(x)| \leq 1\} = \{x_i\}$, $i = 1, \dots, m$, where $x_i < x_{i+1}$. Set $x_{m+1} = b$. Let k_i be the integer such that $2^{-k_i-1} < |h(x_i)| \leq 2^{-k_i}$.

Now let $f_n(x) \equiv 0$ for $x \in [a, x_1]$, $f_n(x_i) = 0$ for $i = 1, \dots, m+1$, and let $f_n(x)$ be linear on $(x_i, x_{i+1}]$, $i = 1, \dots, m$, with $f_n(x_i+) = h(x_i) \cdot 2^{-(n-k_i)}$.

It is easily seen that $\sum f_n(x)$ converges uniformly to a regulated function f such that $f^* = h$.

In the sequel "an interval" means "an interval with more than one point", and I denotes a subinterval of $[a, b]$. For $h: I \rightarrow R$ let $h_+(x) = \max\{h(x), 0\}$, $h_-(x) = \max\{-h(x), 0\}$. The sums $\sum_{x \in I} h_+(x)$, $\sum_{x \in I} h_-(x)$ may be finite or infinite.

If both are infinite we shall say that f is of type $(1, 1)$ on I , if both are finite we shall say that f is of type $(0, 0)$ on I , if the first is infinite and the second finite we shall say that f is of type $(1, 0)$ on I , and if the first is finite and the second infinite we shall say that f is of type $(0, 1)$ on I .

Clearly, if f is of type $(1, 0)$ or $(0, 1)$ on a subinterval then f has no representation $(*)$. We shall show that if a regulated function is of type $(1, 1)$ on every interval, then it has a representation $(*)$. First we shall show that there exist regulated functions which are of type $(1, 1)$ on every interval and that there exist regulated functions which do not have a representation $(*)$, but are of type $(1, 1)$ or $(0, 0)$ on each interval.

EXAMPLE 1. *Regulated functions which are of type $(1, 1)$ on each subinterval J of $[0, 1]$.*

By Theorem 2, we need only find an $h: [0, 1] \rightarrow R$ satisfying the conditions of that theorem such that

$$\sum_{x \in J} h_+(x) = \sum_{x \in J} h_-(x) = \infty \quad \text{for every subinterval } J \text{ of } [0, 1].$$

Let

$$h\left(\frac{2p+1}{2^n}\right) = \frac{(-1)^n}{n}, \quad p = 0, \dots, 2^{n-1} - 1; \quad n = 1, 2, \dots$$

and $h(x) = 0$ otherwise.

EXAMPLE 2. *A regulated function which does not have a representation (*) and is of type (0, 0) or type (1, 1) on every interval.*

Let $\{I_n\}$ be a sequence of nonoverlapping intervals with I_n to the left of I_{n+1} and $\cup I_n = [0, 1)$. On I_{2k-1} , $k = 1, 2, \dots$, let $h(x) = 1/k$ at k^2 interior points and be zero elsewhere. On I_{2k} , $k = 1, 2, \dots$, define a function $\bar{h}(x)$ in a manner analogous to the construction in Example 1 and then set $h(x) = \bar{h}(x)/k$. Then from Theorem 2, there exists a regulated f such that $f^* = h$. On any interval which intersects the interior of an I_{2k} , f is of type (1, 1); on other intervals it is of type (0, 0). If f had the representation (*), then the sum of the jumps would be bounded, but this sum has increment k on I_{2k-1} .

We turn now to our final result.

THEOREM 3. *If f is regulated on $[a, b]$ and is of type (1, 1) on every subinterval, then f has a representation (*).*

PROOF. Suppose f is of type (1, 1) on every subinterval of $[a, b]$ and $M = \max |f^*(x)|$. Let $S = \{x: f^*(x) \neq 0\}$. We begin by showing that given $\varepsilon > 0$ and $\eta > 0$ there is a finite set $F \subset S$, $F = \{\alpha_1, \dots, \alpha_r\}$, where $\alpha_1 < \dots < \alpha_r$, such that

$$\sum_{x \in F} J_x^{f^*(x)} = \sum_{i=1}^r J_{\alpha_i}^{f^*(\alpha_i)}$$

satisfies

$$(i) \quad \left| \sum_{i=1}^k J_{\alpha_i}^{f^*(\alpha_i)}(x) \right| \leq M + \varepsilon \quad \text{for } 1 \leq k \leq r, x \in [a, b],$$

and

$$(ii) \quad \max\{|f^*(x)|: x \in S \setminus F\} < \eta.$$

To show this, let $F_1 = \{a_1, \dots, a_s\}$, where $a_i \in S$, $a < a_1 < \dots < a_s < b$ and $\max\{|f^*(x)|: x \in S \setminus F_1\} < \eta$. For each $i = 1, \dots, s-1$, choose $\{a_{ij}\}$, a finite subset of S , $a_i < a_{i1} < \dots < a_{in_i} < a_{i+1}$, such that the sign of f^* at each a_{ij} is opposite to its sign at a_i and $|f^*(a_i) + \sum_{j=1}^{n_i} f^*(a_{ij})| < \varepsilon/S$. Let F be the set $a_1 < a_{11} < \dots < a_{1n_1} < a_2 < a_{21} < \dots < a_{s-1} < \dots < a_s$. The function $\sum_{x \in F} J_x^{f^*(x)}$ has property (i) and F has property (ii).

We now show that a regulated function f of type (1, 1) on every subinterval has a representation (*). We assign indices to the elements of S in the course of our construction, rather than rearranging a given index set.

Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\sum \varepsilon_n < \infty$. Let $T_1 = \{x: |f^*(x)| \geq 2^{-1}\}$ and, for $n = 2, 3, \dots$, let $T_n = \{x: 2^{-n-1} \leq |f^*(x)| < 2^{-n}\}$. Applying the above remark to f with $\varepsilon = \varepsilon_1$ and $\eta = 2^{-1}$, we obtain an ordered set $S_1 = \{a_i\}$, $i = 1, \dots, i_1$, with $a_i < a_j$ for $i < j$ and $S_1 \supset T_1$. Let

$$f_1 = f - \sum_{x \in S_1} J_x^{f^*(x)}.$$

Now $\max |f_1^*(x)| < 2^{-1}$. Applying the above remark to f_1 with $\varepsilon = \varepsilon_2$ and $\eta = 2^{-2}$, we obtain set $S_2 = \{a_i\}$, $i = i_1 + 1, \dots, i_2$, with $a_i < a_j$ for $i < j$, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 \supset T_1 \cup T_2$. Let

$$f_2 = f_1 - \sum_{x \in S_2} J_x^{f^*(x)}.$$

Having obtained f_n , where n is a positive integer and $\max |f_n^*(x)| < 2^{-n}$, and a finite set S_n with $S_n \cap S_j = \emptyset$, $j < n$, and $S_1 \cup \dots \cup S_n \supset T_1 \cup \dots \cup T_n$, we apply the above remark to f_n with $\varepsilon = \varepsilon_{n+1}$ and $\eta = 2^{-(n+1)}$ to obtain the set $S_{n+1} = \{a_{i_{n+1}}, \dots, a_{i_{n+1}+1}\}$, with $a_i < a_j$ for $i < j$, $S_{n+1} \cap (\cup_1^n S_j) = \emptyset$, $S_{n+1} \supset T_{n+1}$. Let

$$f_{n+1} = f_n - \sum_{x \in S_{n+1}} J_x^{f^*(x)} = f - \sum_{i=1}^{i_{n+1}} J_{a_i}^{f^*(a_i)}.$$

The saltus of f_n at x is less than 2^{-n} for each $x \in [a, b]$ and the sequence $\{f_n\}$ converges uniformly to a function g which is, therefore, continuous. Then

$$f = g + \sum_{i=1}^{\infty} J_{a_i}^{f^*(a_i)}$$

is the desired representation.

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