ON UNIVALENT POLYNOMIALS

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ABSTRACT. We define $V_n \subseteq \mathbb{C}^{n-1}$ to be the set of $(n-1)$ -tuples $(a_2, \dots, a_n)$ such that the polynomial $p(z) = z + a_2 z^2 + \dots + a_n z^n$ is univalent, i.e., one-to-one in $|z| < 1$. In this paper we construct a real polynomial h of degree $4(2(n-1)^2 - 1)(n-1)$ such that if $(a_2, \dots, a_n)$ is in the boundary of V_n then $h(\operatorname{Re} a_2, \operatorname{Im} a_2, \dots, \operatorname{Re} a_n, \operatorname{Im} a_n) = 0$. This shows that the boundary of V_n is a subset of an algebraic submanifold of $\mathbb{R}^{2(n-1)}$.

We define $V_n \subseteq \mathbb{C}^{n-1}$ to be the set of $(n-1)$ -tuples $(a_2, \dots, a_n)$ such that the polynomial $p(z) = z + a_2 z^2 + \dots + a_n z^n$ is univalent, i.e., one-to-one in $|z| < 1$. V_n may also be considered as a subset of $\mathbb{R}^{2(n-1)}$. A convenient description of V_n would be as a region bounded by algebraic submanifolds of $\mathbb{R}^{2(n-1)}$. In Brannan [1], however, V_3 is described as the intersection of regions of this sort. In this paper we construct a real polynomial h of degree $4(2(n-1)^2 - 1)(n-1)$ such that if $(a_2, \dots, a_n)$ is in the boundary of V_n then $h(\operatorname{Re} a_2, \operatorname{Im} a_2, \dots, \operatorname{Re} a_n, \operatorname{Im} a_n) = 0$. This shows that the boundary of V_n is a subset of an algebraic submanifold of $\mathbb{R}^{2(n-1)}$.

In the following, let $p(z) = a_1 z + a_2 z^2 + \dots + a_n z^n$, where $a_1 = 1$.

DEFINITION. We say w is a *vertex* of the curve $p(e^{i\phi})$, $0 \leq \phi \leq 2\pi$, if $p(x) = p(y) = w$ for $x \neq y$, $|x| = |y| = 1$. We say w is a *cusp* of the same curve if $p(x) = w$ and $p'(x) = 0$ for $|x| = 1$.

THEOREM. *The point $(a_2, \dots, a_n)$ is in the interior of V_n if and only if the curve $p(e^{i\phi})$ has no vertices or cusps.*

The proof of this theorem is straightforward. See Quine [3].

To study the condition algebraically we follow a procedure similar to that of Koessler [2]. Define

$$G(x, y) = \begin{cases} (p(x) - p(y))/(x - y) & \text{when } x \neq y, \\ p'(x) & \text{when } x = y. \end{cases}$$

A vertex or cusp of $p(e^{i\phi})$ corresponds to a pair (x_0, y_0) such that $|x_0| = |y_0| = 1$ and $G(x_0, y_0) = 0$. Elementary computations give

$$G(x, y) = \sum_{k=0}^{n-1} b_k(x) y^k$$

where

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$$b_k(x) = \sum_{j=0}^{n-k-1} a_{j+k+1} x^j.$$

Now define

$$G^*(x, y) = x^{n-1} y^{n-1} \overline{G(1/\bar{x}, 1/\bar{y})}.$$

We have

$$G^*(x, y) = \sum_{k=0}^{n-1} C_k(x) y^k$$

where

$$C_k(x) = \sum_{j=n-k-1}^{n-1} \bar{a}_{2(n-1)-(j+k)+1} x^j.$$

From the definition of G^* we see that if $|x_0| = |y_0| = 1$ then $G(x_0, y_0) = 0 \Leftrightarrow G^*(x_0, y_0) = 0$. Now let $R(x)$ be the resultant of $G(x, y)$ and $G^*(x, y)$ as polynomials in y , that is, as polynomials in $\mathbb{C}[x][y]$. (See van der Waerden [4, p. 83].)

THEOREM 2. *The point $(a_2, \dots, a_n)$ is in the interior of V_n if and only if $R(x) \neq 0$ for $|x| = 1$.*

PROOF. If w is a vertex or a cusp of $p(e^{i\phi})$ then we can find x_0 and y_0 such that $|x_0| = |y_0| = 1$, $p(x_0) = w$ and $G(x_0, y_0) = G^*(x_0, y_0) = 0$. From the basic properties of the resultant (see van der Waerden [4]) and by symmetry, $R(x_0) = R(y_0) = 0$. Thus the preimages on $|z| = 1$ of the vertices and cusps of $p(e^{i\phi})$ are among the zeros of $R(x)$. Thus if $R(x) \neq 0$ for $|x| = 1$, then by Theorem 1, $(a_2, \dots, a_n)$ is in the interior of V_n .

Conversely, suppose $R(x_0) = 0$ for $|x_0| = 1$. Then we can find y_0 such that $G(x_0, y_0) = G^*(x_0, y_0) = 0$, i.e. $G(x_0, y_0) = G(x_0, 1/\bar{y}_0) = 0$. If $|y_0| = 1$ then $(a_2, \dots, a_n)$ is not in the interior of V_n by Theorem 1. If $|y_0| \neq 1$ then either $|y_0| < 1$ or $|1/\bar{y}_0| < 1$. Since $p(x_0) = p(y_0) = p(1/\bar{y}_0)$, p is not univalent in $|z| < 1$.

Now we investigate $R(x)$ further. The expression for $R(x)$ is a $2(n-1)$ by $2(n-1)$ determinant whose only entries are 0, $c_k(x)$, and $b_k(x)$, $k = 1, \dots, n-1$. The formal degrees of $b_k(x)$ and $c_k(x)$ in x are $n-k-1$ and $n-1$ respectively. Thus the term in the determinant expression for $R(x)$ with the highest formal degree is the term

$$\begin{aligned} & (b_0(x))^{n-1} (c_{n-1}(x))^{n-1} \\ &= (a_1 + a_2 x + \dots + a_n x^{n-1})^{n-1} (\bar{a}_n + \bar{a}_{n-1} x + \dots + \bar{a}_1 x^{n-1})^{n-1} \\ &= (a_n \bar{a}_1)^{n-1} x^{2(n-1)^2} + \dots + (\bar{a}_n a_1)^{n-1}. \end{aligned}$$

The formal degree of R in x is $2(n-1)^2$ and this is the actual degree if $a_n \neq 0$. We see likewise that the degree of R in the variables $a_2, \bar{a}_2, \dots, a_n, \bar{a}_n$ jointly is $2(n-1)$, if $n > 2$.

We also note that $G(x, y) = G^*(x, y) = 0$ if and only if $G(1/\bar{x}, 1/\bar{y})$

$= G^*(1/\bar{x}, 1/\bar{y}) = 0$. Thus $R(x) = 0$ if and only if $R(1/\bar{x}) = 0$, that is, the zeros of $R(x)$ are symmetric in the circle $|z| = 1$.

THEOREM 3. *If $(a_2, \dots, a_n)$ is on the boundary of V_n , and $p(x_0)$, $|x_0| = 1$, is a vertex or a cusp of $p(e^{i\phi})$, then x_0 is a double zero of $R(x)$.*

PROOF. Consider R as a function of $(a_2, \dots, a_n)$ and of x . Take $(\alpha_2, \dots, \alpha_n)$ in the interior of V_n near $(a_2, \dots, a_n)$. Then $R(\alpha_2, \dots, \alpha_n, x)$ has a zero x_1 near x_0 and $|x_1| \neq 1$ by Theorem 2. Since $1/\bar{x}_1$ is also a zero, $R(\alpha_2, \dots, \alpha_n, x)$ has two zeros near x_0 . Therefore x_0 is a double zero of $R(a_2, \dots, a_n, x)$.

Now let $N = 2(n-1)^2$, $S(x) = R'(x)$, $S^*(x) = x^{N-1} \overline{S(1/\bar{x})}$. Let $Q = Q(a_2, \bar{a}_2, \dots, a_n, \bar{a}_n)$ be the resultant of S and S^* . The determinant expression for Q will be a $2(N-1)$ by $2(N-1)$ determinant whose only entries are 0 and the coefficients of S and S^* . If $n > 2$, each of these coefficients is of degree $2(n-1)$ in $(a_2, \bar{a}_2, \dots, a_n, \bar{a}_n)$, so the degree of Q is $4(N-1)(n-1)$ in $(a_2, \bar{a}_2, \dots, a_n, \bar{a}_n)$.

THEOREM 4. *If $(a_2, \dots, a_n)$ is on the boundary of V_n then $Q(a_2, \bar{a}_2, \dots, a_n, \bar{a}_n) = 0$.*

PROOF. By Theorem 3, $R(x)$ had a double zero x_0 , with $|x_0| = 1$. Therefore $R'(x_0) = S(x_0) = S^*(x_0) = 0$. Thus S and S^* have a common root and their resultant is zero.

To show that $Q = 0$ defines an algebraic submanifold of R^{2n} , we show that Q is real so that as a polynomial in $\text{Re } a_2, \text{Im } a_2, \dots, \text{Re } a_n, \text{Im } a_n$, Q will have real coefficients. Let $S(x) = Ax^{N-1} + \dots + B$. Then $S^*(x) = \bar{B}x^{N-1} + \dots + \bar{A}$. Without loss of generality, assume that $a_n \neq 0$ so that $A = N(a_n a_1)^{n-1}$ is not zero. Let $x_j, j = 1, \dots, N-1$, be the zeros of S . Then $1/\bar{x}_j, j = 1, \dots, N-1$, are the zeros of S^* , and

$$Q = (A \bar{B})^{N-1} \prod_{j,k=1}^{N-1} (x_j - 1/\bar{x}_k).$$

See van der Waerden [4, p. 87]. Therefore

$$Q = |A|^{2(N-1)} \prod_k (1 - |x_k|^2) \prod_{j < k} |1 - x_j \bar{x}_k|^2$$

since $B/A = \prod_k x_k$. Thus Q is real.

Summarizing, we have

THEOREM 5. *If $n > 2$, the boundary of V_n is a subset of an algebraic submanifold of $R^{2(n-1)}$ given by a polynomial of degree $4(2(n-1)^2 - 1)(n-1)$.*

Let us look at the case $n = 3$. This has been investigated by Brannan [1]. There is no loss in generality in taking a_3 real and positive. Thus letting $a_2 = \xi + i\eta$, $a_3 = \zeta > 0$, Brannan showed that $(a_2, a_3) \in V_3$ if and only if $f(\xi, \eta, \zeta, d) \leq 0$ for $-1 \leq d \leq 3$ where

$$f(\xi, \eta, \zeta, d) = (1 + d)\{\xi^2/(1 + \zeta d)^2 + \eta^2/(1 - \zeta d)^2\} - 1.$$

This shows that the point (ξ, η) belongs to the closed interior of a family of ellipses for fixed ζ . If $0 \leq a_3 \leq 1/5$, the points on the boundary of V_3 satisfy

$4\{\xi^2/(1+3\xi)^2 + \eta^2/(1-3\xi)^2\} = 1$. On the part of the boundary where $1/5 \leq a_3 \leq 1/3$, a portion of the boundary is in the envelope of the above family of ellipses. Now f of degree 4 in d and 4 in (ξ, η, ζ) jointly. Using the resultant to eliminate d from f and $\partial f/\partial d$ would give us an equation $R(\xi, \eta, \zeta) = 0$ where R is at most of degree $(4+3)4 = 28$. Our $Q(a_2, \bar{a}_2, a_3, \bar{a}_3)$ is of degree 56. We remark that our method is essentially the same as Brannan's except that Brannan makes the change of variables $x + y = 2zc$, $xy = z^2$ and then eliminates z using Cohn's rule instead of the resultant. The fact that the degree of the boundary under Brannan's computations is half that predicted by Theorem 5 is due to his substituting along the way for c^2 . At any rate, it appears that the boundary of V_n has an equation of high degree even for small n .

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