PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 1, May 1976

NONATOMIC BANACH LATTICES CAN HAVE l_1 AS A DUAL SPACE

E. LACEY AND P. WOJTASZCZYK

ABSTRACT. Examples of nonatomic M spaces whose duals are l_1 are constructed.

In the theory of (separable) Banach lattices it is sometimes true that a space X is purely atomic if and only if X^* is (for example, if X is order complete and X^* is separable). However, in general, it is not true. The main purpose of this paper is to illustrate this with the following theorem on M spaces.

THEOREM 1. *Let α be a countable ordinal ($\alpha \geqslant \omega$). Then there is a nonatomic M space X whose dual is linearly order isometric to l_1 and such that X is linearly isomorphic to $C\langle 1,\alpha\rangle$.*

Unexplained terminology shall be that of [5]. By an *atom* in a Banach lattice we shall mean an element $x > 0$ such that if $0 \leqslant y \leqslant x$, then $y = ax$ for some $a \geqslant 0$. A *purely atomic* Banach lattice is one in which every positive element dominates some atom and a *nonatomic* Banach lattice is one which admits no atoms. By $\langle 1, \alpha\rangle$ we mean the order interval of ordinals between 1 and α with the usual order topology.

The following theorem is known. We state and prove it only as background to the discussion.

THEOREM 2. *Let X be a separable order complete Banach lattice.*
(a) *If X is purely atomic and X^* is separable, then X^* is purely atomic*;
(b) *If X^* is purely atomic, then so is X.*

PROOF. (a) Since X is separable and order complete, the norm is order continuous (see [7]). Thus the atoms form an unconditional basis for X and the sequence of corresponding biorthogonal functionals (which are atoms) in X^* forms an unconditional basis for X^* (see [3]).

(b) Suppose that X^* is purely atomic and let $x^* \in X^*$ be a nonzero lattice homomorphism (i.e. an atom). Then the kernel I of x^* is a proper closed ideal in X and since X has order continuous norm, I is a band and $X = I \oplus I^\perp$. Clearly $I^\perp = \operatorname{span}(x)$ for some $x > 0$ and x is an atom in X. Now suppose $y > 0$ and $z^*(y) = 0$ for all atoms z^* in X^*. Then for $F = \{y\}^{\perp\perp}$, $z^*|F = 0$ for all atoms z^*. Let $y^* > 0$ be such that $y^*|F \neq 0$ and $y^*|F^\perp = 0$. Then $y^* \wedge z^* = 0$ for all atoms z^*, which is impossible. Thus X is purely atomic.

A simple example shows that order completeness is necessary for the result

Received by the editors January 9, 1975 and, in revised form, March 12, 1975.
AMS (MOS) subject classifications (1970). Primary 46E05; Secondary 46E15.
Key words and phrases. M space, nonatomic and purely atomic Banach lattices.

in (a) above. For each n and k with $0 \leqslant k < 2^n$ let $I_{n,k} = (k/2^n, (k+1)/2^n)$ and put X_n to be the span of the characteristic functions of $\{I_{n,k}: 0 \leqslant k < 2^n\}$. We define X by

$$X = \left\{ \{f_n\}: f_n \in X_n \quad \text{and} \lim_{n\to\infty} f_n = f \in L_p[0,1] \right\}$$

where the limit is taken in the L^p-norm. Clearly X is a linear space and, in fact, a Banach lattice under the supremum norm given by $\|\{f_n\}\| = \sup_n \|f_n\|^p$, and the positive cone $C = \{\{f_n\}: f_n \geqslant 0 \text{ for all } n\}$. A simple calculation shows that X^* is linearly order isomorphic to $[(\otimes \sum_{n=1}^{\infty} l_q(2^n))_{l_1} \otimes L_q[0,1]]_{l_1}$ which is clearly not purely atomic since $L_q[0,1]$ is atomless $(1/p + 1/q = 1)$. But, each of the characteristic functions $\chi_{I_{n,k}}$ generates an atom in X by defining $f_m = 0$ if $m \neq n$ and $f_n = \chi_{I_{n,k}}$ and it is trivial to see that X is purely atomic.

The proof of (b) does not really depend on separability. If X is purely atomic and order complete, then X cannot contain a sublattice isomorphic to l_∞. Since, otherwise, X^* contains one isomorphic to l_∞^* and, hence, X^* contains a sublattice isomorphic to $L_1[0,1]$, which is impossible if X^* is purely atomic since $L_1[0,1]^* = L_\infty[0,1]$ is also atomless. (Also, statement (b) was mentioned without proof in [6] where the author inadvertently left off the hypothesis of order completeness.)

Our initial M space will be formally constructed from a set of relations on $C\langle 1, \omega^\omega\rangle$. However, it is easier to visualize a description of it by first building a certain union of intervals in the plane and taking a subspace of the space of continuous functions on this space (we wish to thank M. Starbird for suggesting the form of this set to us). For simplicity we give a heuristic description of the space. A formal construction is possible, but nothing seems to be gained by insisting on such a completely formal description. Each interval we consider shall be a closed interval of a certain length less than or equal to one. We shall assume that all of these intervals lie in the first quadrant of the plane and that they start at the origin (so, they are determined by their length and the angle they make with the positive x-axis). Let $\{d_n\}$ be a strictly decreasing sequence in $(0,1]$ with $d_0 = 1$. We put I_0 to be the unit interval along the positive y-axis. Next take a sequence of intervals of length d_1 whose slopes are strictly increasing to $+\infty$ (i.e. the sequence converges to I_0). For example, we may label this sequence as $\{I_{0,n}\}$ where the slope of $I_{0,n+1}$ is greater than the slope of $I_{0,n}$. The idea is that to each interval $I_{0,n}$ we now correspond yet another sequence of intervals, each of length d_2, whose slopes are strictly increasing to the slope of $I_{0,n}$ and such that if $n > 1$, then all of these intervals lie strictly between $I_{0,n-1}$ and $I_{0,n}$. We continue in this manner using as the next length the number d_3 and so on. This construction is such that for each arc S_n in the first quadrant of the circle of radius d_n centered at the origin, S_n intersects this class of intervals in a set homeomorphic to $\langle 1, \omega^n\rangle$ where, in fact, we can take the natural order on this intersection as we progress in the counter-clockwise direction from 0 to $\pi/2$. We let T be the compact space which is the union of all of these intervals. It is easy to see that the closure of the end-points of these intervals in T is naturally homeomorphic to $\langle 1, \omega^\omega\rangle$. We associate an M space with T in the following natural way: $X = \{f \in C(T): f(0,0) = 0$ and f is affine when restricted to any interval in $T\}$.

Clearly X is indeed an M space. Moreover, since each $f \in X$ is affine when restricted to an interval, it is completely determined on this interval by its value at the end-points. Thus X can be embedded in a linear order isometric fashion into $C\langle 1, \omega^\omega \rangle$ simply by taking the restriction operator $f \to f|T_0$ where T_0 is the closure of the end-points of the intervals in T. One can easily see that X is then just the space of functions in $C\langle 1, \omega^\omega \rangle$ satisfying a certain set of relations, e.g., $f \in X$ satisfies $f(0, d_1) = d_1 f(0, 1)$ among other relations. Now $C\langle 1, \omega^\omega \rangle^* \cong l_1$ and, hence, it follows readily that $X^* \cong l_1$ (see [5]), that is, X^* is linearly order isometric to l_1. Moreover, from the definition of X it follows by easy calculation that X is nonatomic. So we have now constructed a nonatomic M space whose dual is equivalent to l^1, a purely atomic Banach lattice. We focus the remainder of the paper on the question of identifying X.

A theorem of Benyamini says that a separable M space is linearly isomorphic to a space of continuous functions (see [1]). Also, Bessaga and Pełczyński have classified the isomorphic classes of spaces of the type $C\langle 1, \alpha \rangle$ where α is a countable ordinal as follows: if $\omega \leqslant \alpha < \beta < \Omega$, then $C\langle 1, \alpha \rangle$ is linearly isomorphic to $C\langle 1, \beta \rangle$ if and only if $\beta < \alpha^\omega$ (see [2]). Thus, combining these two theorems we see that the space X above is linearly isomorphic to either c_0 or $C\langle 1, \omega^\omega \rangle$. We shall show that both choices are possible depending on the nature of the sequence $\{d_n\}$ chosen to construct X. There is a natural linear isometric embedding of $C\langle 1, \omega^n \rangle$ into $C(S_n)$ which is given as follows: for $f \in C\langle 1, \omega^n \rangle$, $\Phi_n f | S_n \cap T = f$ (we consider that $\langle 1, \omega^n \rangle = S_n \cap T$ where S_n is the arc of radius d_n). If $s \in S_n \backslash T$ and s lies between s_j and s_k in $S_n \cap T$ and s_j, s_k are the closest members of $S_n \cap T$ to s, let r_0 be the unique element on the chord between s_j and s_k determined by drawing the line from origin to s and putting r_0 to be the point of intersection of this line with the chord. Thus $r_0 = as_j + (1 - a)s_k$ where $0 < a < 1$ and we define $(\Phi_n f)(s) = af(s_j) + (1 - a) f(s_k)$. It follows readily that $\Phi_n(f) \in C(S_n)$ and that Φ_n is a linear isometry of extension. We let $Q_n \colon X \to X$ be the operator defined by

$$(Q_n f)(t) = (|t|/d_n)\Phi_n(f|S_n \cap T)(d_n t/|t|).$$

Since Φ_n is an extension operator, $Q_n \circ Q_n = Q_n$, i.e., Q_n is a projection on X.

LEMMA 1. $\|Q_n\| = 1$.

PROOF. Let $f \in X$ with $\|f\| = 1$ and suppose $t \in T$ is in an interval of length $\geqslant d_n$. Then $d_n t/|t| \in S_n \cap T$ and

$$|(Q_n f)(t)| = |(|t|/d_n) f(d_n t/|t|)| = |f(t)| \leqslant 1.$$

If t is in an interval of T of length $< d_n$, then since $|\Phi(f|S_n \cap T)(d_n t/|t|)| \leqslant 1$ and $|t|/d_n \leqslant 1$, $|(Q_n f)(t)| \leqslant 1$.

LEMMA 2. *For any* $f \in X$, $\lim_{n\to\infty} \|Q_n f - f\| = 0$.

PROOF. We observed in the proof of Lemma 1 that if $t \in T$ and t is in an interval of length $\geqslant d_n$, then $(Q_n f)(t) = f(t)$ and if t is in an interval of length $< d_n$, $|(Q_n f)(t)| \leqslant \|f|S_n \cap T\|$. Let $f \in X$ and $\varepsilon > 0$ be given. Let N be such that if $|t| \leqslant d_N$, then $|f(t)| < \varepsilon$. For $n > N$,

$$|(Q_n f)(t) - f(t)| = \begin{cases} 0 & \text{if } t \text{ is in an interval of length} \geqslant d_n, \\ \leqslant 2\|f|S_n \cap T\| \leqslant 2\varepsilon & \text{otherwise.} \end{cases}$$

LEMMA 3. $Q_n \circ Q_m = Q_m \circ Q_n = Q_{\min(n,m)}$.

The proof of this is immediate.

If we put $Q_0 = 0$, then Lemma 2 above shows that $X = \sum_{n=1}^{\infty} (Q_n - Q_{n-1})(X)$. Let $X_n = (Q_n - Q_{n-1})(X) = \{f \in X\colon Q_n f = f \text{ and } Q_{n-1} f = 0\}$. A routine calculation shows that X_n is linearly isometric to $(c_0 \oplus \cdots \oplus c_0)_{l_\infty} = c_0$, where there are n summands in the direct sum. Hence if we can put a condition on the sequence $\{d_n\}$ which implies that $\sum_{n=1}^{\infty} f_n$ converges in norm whenever $f_n \in X_n$ with $\|f_n\| \to 0$, then we shall have that X is linearly isomorphic to $(\oplus \sum_{n=1}^{\infty} X_n)c_0$ which is, in turn, linearly isomorphic to c_0 by the above remarks.

We now show that $d_n = \lambda^n$ where λ is a fixed number with $0 < \lambda < 1$ defines such a sequence. Suppose that $f_n \in X_n$ and $\lim_{n\to\infty} \|f_n\| = 0$. For $\varepsilon > 0$ let N be chosen such that if $n \geqslant N$, then $\max\{\|f_n\|, d_n\} < \varepsilon$. Then for $n \geqslant N$, f_n is identically zero on intervals of length greater than d_n. If $m > N$, then

$$\left\| \sum_{i=m}^{m+k} f_i \right\| = \sup_{|t| < d_m} \left| \sum_{i=m}^{m+k} f_i(t) \right|.$$

Moreover, if t is in an interval of length d_s for some $s < m$, then the right-hand side of the equality is 0.

If t is in an interval of length d_s for some $s > m$, then

$$\left| \sum_{i=m}^{m+k} f_i(t) \right| \leqslant \sum_{i=m}^{s} |f_i(t)| < \sum_{i=m}^{s} \|f_i\| \frac{d_s}{d_i} = \sum_{i=m}^{s} \|f_i\| \frac{\lambda^s}{\lambda^i}$$
$$\leqslant \varepsilon \sum_{i=1}^{s} \lambda^{s-i} \leqslant \varepsilon \sum_{i=0}^{\infty} \lambda^i = \frac{\varepsilon\lambda}{1-\lambda}.$$

We note that the above result can be deduced from an unpublished refinement of Benyamini's theorem due to A. Gleit [4]. Namely, by Gleit's result, if $\sup_n d_{n+1}/d_n < 1$, then the space X above is linearly isomorphic to c_0.

To show that for certain sequences $\{d_n\}$ the space X constructed above can be linearly isomorphic to $C\langle 1, \omega^\omega \rangle$ we need only show that there are sequences $\{d_n\}$ for which the corresponding X is not linearly isomorphic to c_0.

Recall that the Banach-Mazur distance between two Banach spaces X and Y is the infimum, $d(X, Y)$, of $\|T\|\,\|T^{-1}\|$ where T ranges over all invertible operators from X onto Y ($d(X, Y) = \infty$ if the spaces are not linearly isomorphic).

From the Bessaga-Pełczyński theorem cited above $Q_n = d(c_0, C\langle 1, \omega^n \rangle)$ is a finite number. Moreover $\{a_n\}$ is increasing. Clearly $\lim_{n\to\infty} a_n = \infty$ since $(\oplus \sum_{n=1}^{\infty} C\langle 1, \omega^n \rangle)_{c_0}$ is linearly isomorphic to $C\langle 1, \omega^\omega \rangle$ and if $\lim_{n\to\infty} a_n < \infty$, then $(\oplus \sum_{n=1}^{\infty} C\langle 1, \omega^n \rangle)_{c_0}$ would also be linearly isomorphic to c_0.

From the definition of Q_n, $Q_n(X)$ is easily seen to be linearly isomorphic to $C\langle 1, \omega^n \rangle$ and, in fact, $d(Q_n(X), C\langle 1, \omega^n \rangle) \leqslant 1/d_n$.

Finally we recall the well-known result of Pełczyński that a complemented infinite dimensional subspace of c_0 is linearly isomorphic to c_0 (see [5]). Furthermore, for each $\lambda \geqslant 1$, the supremum, K_λ, of $d(c_0, P(C_0))$ where P ranges over all projections on c_0 with infinite-dimensional range and such that $\|P\| \leqslant \lambda$ is finite. For, suppose there is a sequence of such projections P_n such that $d(c_0, P_n(c_0)) \geqslant n$. Since $\|P_n\| \leqslant \lambda$ for all n, $(\oplus \sum_{n=1}^{\infty} P_n(c_0))_{c_0}$ is complemented in $(\oplus \sum_{n=1}^{\infty} c_0)_{c_0} = c_0$, but cannot be isomorphic to c_0 since $\lim_{n\to\infty} d(c_0, P_n(c_0)) = \infty$.

We now show that if the sequence $\{d_n\}$ is chosen such that $\{a_n d_n\}$ is unbounded, then the associated space X is not linearly isomorphic to c_0. For, suppose T is an invertible operator of X onto c_0 and put $\lambda = \|T\|\,\|T^{-1}\|$. Then for each n, $TQ_nT^{-1} = P_n$ is a projection on c_0 with infinite-dimensional range and $\|P_n\| \leqslant \lambda$. Thus

$$\begin{aligned} a_n &= d(c_0, C\langle 1, \omega^n\rangle) \\ &\leqslant d(c_0, P_n(c_0)) \cdot d(P_n(c_0), Q_n(X)) \cdot d(Q_n(X), C\langle 1, \omega^n\rangle) \\ &\leqslant K_\lambda \cdot \lambda / d_n, \end{aligned}$$

where K_λ is the constant given above for λ. This contradicts the choice of the sequence $\{d_n\}$ since $\{a_n d_n\}$ is bounded by λK_λ, which is a fixed constant.

Recall that if S is a compact Hausdorff space and Y is a Banach space, then the space $C(S, Y)$ of all Y-valued continuous functions on S is a Banach space in the supremum norm. Moreover, if $Z_n = C(S)$ for each n, then $(\oplus \sum_{n=1}^{\infty} Z_n)_{c_0}$ is linearly isomorphic to $C(S, c_0)$. In particular, if α is a countable ordinal, then $C\langle 1, \alpha\rangle$ is linearly isomorphic to $C(\langle 1, \alpha\rangle, c_0)$ since $C\langle 1, \alpha\rangle$ is linearly isomorphic to $(\oplus \sum_{n=1}^{\infty} Z_n)_{c_0}$ where $Z_n = C\langle 1, \alpha\rangle$ (see [5]).

Proof of Theorem 1. By the above we can construct a nonatomic M space X which is linearly isomorphic to c_0. It is easy to see that $C(\langle 1, \alpha\rangle, X)$ is an M space under pointwise ordering. Now $C(\langle 1, \alpha\rangle, X)$ is clearly linearly isomorphic to $C(\langle 1, \alpha\rangle, c_0)$ which, in turn, is linearly isomorphic to $C\langle 1, \alpha\rangle$ by the above remarks.

Finally we note that it is possible for a nonatomic Banach lattice Y to have the property that Y^* has exactly n atoms ($n = 1, 2, \cdots$). Clearly we need only demonstrate that there is such a Y so that Y^* has exactly one atom. We can do this in a manner similar to the example given after Theorem 2 above. Both of these examples are motivated by work in [8]. Let $1 < p < \infty$ and $f \in L_p[0,1]$ be a fixed positive element of norm one. Then $Y = \{\{f_n\}: f_n \in L_p[0,1]$ and $\{f_n\}$ converges to af for some scalar $a\}$. Then Y is a nonatomic Banach lattice under the supremum norm and coordinatewise ordering and its dual has exactly one atom.

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Department of Mathematics, University of Texas, Austin, Texas 78712

Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland