ON THE PROBLEMS OF DETERMINING KNOTS BY THEIR COMPLEMENTS AND KNOT COMPLEMENTS BY THEIR GROUPS

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ABSTRACT. Modulo the conjecture that the complement of a prime knot in the 3-sphere is determined by its fundamental group, we show that at most finitely many mutually inequivalent knots can have homeomorphic complements.

In this note, we shall exhibit relationships between the following well-known conjectures about tame knots in the 3-sphere. Our main result is that, modulo the *Complements* conjecture, given four knots with homeomorphic complements, at least two of the knots must be equivalent.

KNOT TYPE CONJECTURE. If K and L are knots with homeomorphic complements, then K and L are equivalent knots (in the sense that there exists a homeomorphism $f: S^3 \rightarrow S^3$ such that $f(K) = L$).

KNOT COMPLEMENTS CONJECTURE. If K and L are prime knots with isomorphic knot groups then $S^3 - K$ and $S^3 - L$ are homeomorphic.

The *Type* conjecture has been established, in a somewhat stronger form, for many knots ([1], [3], [7]–[10], [13], [14]). The *Complements* conjecture became approachable with the advent of the Annulus Theorem and the Torus Theorem [5], [6], [15]. It follows from Theorem 10 of [6], Lemmas 2.1 and 2.2 of [12], and the main result of [2], that if K and L are counterexamples to the *Complements* conjecture, then each is a nontrivial cable knot; using the notation of [12, p. 2], $K = J(p, q; K_1)$ and $L = J(r, s; L_1)$ for some nontrivial knots K_1, L_1 ($|q|, |s| \geq 2$).

For a knot K , let $C^3(K)$ denote the closed complement of a regular neighborhood of K and let (μ_K, λ_K) be a meridian-longitude pair for K . Suppose L is a knot such that $C^3(K)$ is homeomorphic to $C^3(L)$. If $f: C^3(K) \rightarrow C^3(L)$ is a homeomorphism then, for homology reasons, $f(\lambda_K) = \lambda_L^{\pm 1}$ and, for some integer n , $f(\mu_K) = \mu_L^{\pm 1} \lambda_L^n$. If $n = 0$, then f extends to a homeomorphism of S^3 and we can conclude that K and L are equivalent. Modulo the *Complements* conjecture, we can reach the same conclusion if we are given any homeomorphism f with $|n| \neq 1, 2$, or 4.

THEOREM 1 (MODULO THE KNOT COMPLEMENTS CONJECTURE). *If K and L are knots with homeomorphic complements, and there exists a homeomorphism*

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$f: C^3(K) \rightarrow C^3(L)$ such that $f(\mu_K) = \mu_L^{\pm 1} \lambda_L^n$, where $|n| \neq 1, 2$, or 4 , then K and L are equivalent knots.

PROOF. Suppose such a homeomorphism exists. If $n = 0$, then, as noted above, f extends to a homeomorphism of S^3 taking K to L . We assume now that $n \neq 0$. Orient the curves μ_L, λ_L so that $f(\lambda_K) = \lambda_L$ and $f(\mu_K) = \mu_L \lambda_L^n$, $|m| \neq 0, 1, 2$, or 4 .

If J is a (p, q) -curve on $\partial C^3(K)$ (i.e., $J \sim p\mu_K + q\lambda_K$), $f(J)$ is a $(p, mp + q)$ curve on $\partial C^3(L)$. If m is even, let $p = 1$ and $q = -m/2$; if m is odd, let $p = 2$ and $q = -m$. We then have p and q relatively prime, $|q| \geq 3$, and $mp + q = -q$.

Since f maps the curve $\mu_K^p \lambda_K^q$ to the curve $\mu_L^p \lambda_L^{-q}$, the induced isomorphism $f_*: \pi_1(C^3(K)) \rightarrow \pi_1(C^3(L))$ can be extended to an isomorphism of amalgamated free products

$$\pi_1(C^3(K)) \underset{\mu_K^p \lambda_K^q = t^q}{*} \langle t \rangle \cong \pi_1(C^3(L)) \underset{\mu_L^p \lambda_L^{-q} = s^{-q}}{*} \langle s \rangle$$

by sending t to s^{-1} . In other words, the cable knots $J(p, q; K)$ and $J(p, -q; L)$ have isomorphic groups. Assuming the *Complements* conjecture, we then see that $J(p, q; K)$ and $J(p, -q; L)$ have homeomorphic complements. But since $|q| \geq 3$, $|q|$ -strand cable knots have "Property P" [7], [14], so we conclude that $J(p, q; K)$ and $J(p, -q; L)$ are equivalent knots. It follows (see [4, p. 10]) that $J(p, q; K)$ has the same isotopy type as $J(p, -q; L)$ or its mirror image (which is a cable about the mirror image of L) and so (Theorem 21.5 of [11]) K has the same isotopy type as L or its mirror image, i.e., K and L are equivalent.

In all of the references cited earlier, the classes of knots for which the *Type* conjecture holds have been shown to have a stronger property. The following definition of "Property P" is equivalent to the original definition given in [1].

DEFINITION. For a given integer $n \geq 1$, a knot K has *Property P_n* if neither of the manifolds obtained by attaching 2-handles ("pillboxes") to $C^3(K)$ along $(\pm 1, n)$ curves on $\partial C^3(K)$ is simply connected. We say K has *Property P* if K has *Property P_n* for each n .

If it can be shown that all knots have *Property P_n* for $n = 1, 2$, and 4 , then Theorem 1 will say that the *Complements* conjecture implies the *Type* conjecture. In the proof of Theorem 1, we cannot handle the case $|n| = 4$ since it is still unknown whether 2-strand cable knots are determined by their complements. By Theorem 2(c) of [14], *Property P₁* is enough to conclude that cable knots have *Property P*. We thus have the following variation on Theorem 1.

THEOREM 2 (MODULO THE COMPLEMENTS CONJECTURE). *If all nontrivial knots have Property P_n for $n = 1$ and 2 , then the Type conjecture is a theorem.*

Our final result indicates that any proof of the *Complements* conjecture would have to contain a proof of a weak form of the *Type* conjecture, namely that there is an upper bound to the number of mutually inequivalent knots all with the same complement.

THEOREM 3 (MODULO THE COMPLEMENTS CONJECTURE). *If K_i ($i = 1, 2, 3, 4$)*

are knots with homeomorphic complements, then at least two of these knots are equivalent.

PROOF. For $i = 1, 2, 3, 4$, let (μ_i, λ_i) be a meridian-longitude pair for K_i and for $i = 1, 2, 3$, let $f_i: C^3(K_i) \rightarrow C^3(K_{i+1})$ be a homeomorphism. Orient the curves μ_i, λ_i so that for some integers n_i , $f_i(\mu_i) = \mu_{i+1}\lambda_{i+1}^{n_i}$ and $f_i(\lambda_i) = \lambda_{i+1}$ ($i = 1, 2, 3$). For each i, j ($1 \leq i < j \leq 4$), let $g_{ij}: C^3(K_i) \rightarrow C^3(K_j)$ be the homeomorphism obtained by composing the f_i . It is easy to show that there do not exist integers a, b , and c , such that $\{a, b, c, a+b, b+c, a+b+c\} \subseteq \{\pm 1, \pm 2, \pm 4\}$. Therefore, for some $i < j$, g_{ij} satisfies the hypothesis of Theorem 1, and so we can conclude that K_i and K_j are equivalent.

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