ON THE DENSITY CHARACTER OF CLOSED SUBGROUPS

JOHN GINSBURG, M. RAJAGOPALAN AND VICTOR SAKS

ABSTRACT. Answering a question posed by W. W. Comfort, G. L. Itzkowitz, and K. A. Ross, it is shown that for every infinite cardinal number α there are a Hausdorff topological group G and a closed subgroup H of G such that $d(H) > d(G) = \alpha$ (here d denotes "density character"). Specifically, for G we take the (appropriately topologized) free Abelian group $A(\beta D)$ generated by the Stone-Čech compactification of the discrete space D with $|D| = \alpha$.

The principal result. For a space X , the free Abelian group $A(X)$ and its topology are defined as usual (see Markov [M] or Graev [G]; or see [HR, Theorem 8.8] for a detailed exposition of the non-Abelian analogue). In particular we have $A(X) \subset P = \prod_{i \in I} G_i$, where $\{(G_i, f_i) : i \in I\}$ is an enumeration of all topological Abelian groups G_i and continuous functions f_i from X to G_i (with, say, $G_i \subset PP(X)$, this latter condition ensuring that I is a set). We need the following basic properties.

- (i) X is a closed subspace of $A(X)$;
- (ii) algebraically, $A(X)$ is the free Abelian group generated by X ;
- (iii) for every continuous function f from X into a (Hausdorff) Abelian group G there is a continuous homomorphism $\tilde{f} : A(X) \rightarrow G$ such that $f \subset \tilde{f}$.

In what follows we fix the infinite cardinal α ; D and βD are as defined in the abstract. We set $G = A(\beta D)$ and we denote by H the subgroup of G generated (in the group-theoretic sense) by the set $U(\alpha)$ of uniform ultrafilters on α ; these are by definition the ultrafilters $p \in \beta D$ such that $A \in p$ implies $|A| = \alpha$. We use additive notation, so that

$$G = \left\{ \sum_{k=1}^n z_k \cdot p_k : n < \omega, z_k \in Z, p_k \in \beta D \right\}.$$

LEMMA 1. $d(G) = \alpha$.

PROOF. For $n < \omega$, define

$$G_n = (\beta D)^n = \left\{ \sum_{k=1}^n z_k \cdot p_k : z_k \in Z, p_k \in \beta D \right\}.$$

Received by the editors July 9, 1975.

AMS (MOS) subject classifications (1970). Primary 22D05, 54A25; Secondary 22A05.

Key words and phrases. Topological group, free topological group, density character, closed subgroup.

Then $G = \bigcup_{n < \omega} G_n$ and $d(G_n) = \alpha$, so that $d(G) \leq \omega \cdot \alpha = \alpha$.

Let $S \subset G$ be a subset of cardinality $< \alpha$. Let $F \subset \beta D$ be the set of all elements occurring in a minimal representation of some element of S as integral combination of members of βD . Then $|F| < \alpha$. Then $\bar{F} \neq \beta D$. Let $x_0 \in \beta D - \bar{F}$. Then $\exists f: \beta D \rightarrow [0, 1] \ni f$ is continuous and $f(x_0) = 1$ and $f(\bar{F}) = \{0\}$. Let $\tilde{f}$ be its continuous homomorphic extension from $G \rightarrow \pi$. Then $\tilde{f}(S) = \{0\}$ and $\tilde{f}(x_0) = 1$. So $\bar{S} \neq G$. So $d(G) \geq \alpha$.

LEMMA 2. H is a closed subgroup of G .

PROOF. Let $x = \sum_{k=1}^n z_k \cdot p_k \in G \setminus H$; we assume without loss of generality that $p_1 \notin U(\alpha)$, $z_1 \neq 0$, and $p_k \neq p_1$ for $2 \leq k \leq n$. There is a continuous function f from βD to the group of real numbers such that $f(p_1) = 1$ and $f(p_k) = 0$ for $2 \leq k \leq n$ and $f(p) = 0$ for all $p \in U(\alpha)$. Clearly $\tilde{f}(p) = 0$ for all $p \in H$, while $\tilde{f}(x) = z_1 \neq 0$. Thus $x \notin \text{cl}_G H$.

LEMMA 3. $d(H) > \alpha$.

PROOF. Suppose that $E = \{x(\xi): \xi < \gamma\}$ is dense in H with $\omega \leq \gamma \leq \alpha$, say with

$$x(\xi) = \sum_{k=1}^{n(\xi)} z_k(\xi) \cdot p_k(\xi) \quad (n(\xi) < \omega, z_k(\xi) \in Z, p_k(\xi) \in U(\alpha)).$$

We define $S = \{p_k(\xi): \xi < \gamma, k \leq n(\xi)\}$, so that $|S| \leq \omega \cdot \gamma = \gamma$. It is well known (see for example [GJ, Exercise 12B], or [CN, Corollary 12.20]) that the space $U(\alpha)$ admits a family $\mathcal{U}$ of nonempty, pairwise disjoint, open subsets such that $|\mathcal{U}| > \alpha$. It follows that S is not dense in $U(\alpha)$; we choose $p \in U(\alpha) \setminus \text{cl}_{U(\alpha)} S$. As in the proof of Lemma 2 there is a continuous, real-valued function f on $U(\alpha)$ such that $f(p) = 1$ and $f(q) = 0$ for all $q \in \text{cl}_{U(\alpha)} S$. Since $U(\alpha)$ is closed in the normal space βD , the function f extends continuously to a real-valued function g defined on βD and we have $\bar{g}(p) = g(p) = 1$ and $\bar{g}(q) = 0$ for all $q \in S$; since $\bar{g}$ is a homeomorphism, it follows that $\bar{g}(q) = 0$ for all $q \in E$. Since E is dense in H and $p \in H$ we have a contradiction. So the proof is complete.

Finally, the theorem announced in the abstract follows immediately from Lemmas 1, 2 and 3.

REMARK. It is known (see [CIR] and [TSR]) that if H is a closed subgroup of a locally compact group G , then $d(H) \leq d(G)$.

ACKNOWLEDGEMENT. Thanks are due to Professor W. W. Comfort for his kind encouragement and interest that helped this paper to be in this form.

REFERENCES

- [CIR] W. W. Comfort, Gerald L. Itzkowitz and Kenneth A. Ross, *Density character in topological groups* (to appear).
- [CN] W. W. Comfort and S. Negrepointis, *The theory of ultrafilters*, Grundlehren der math. Wissenschaften, vol. 211, Springer-Verlag, Heidelberg, 1974.
- [G] M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948), 279–324; English transl., Amer. Math. Soc. Transl. (1) **8** (1962), 305–364. MR **10**, 11.
- [GJ] Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, University Ser. in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

[HR] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis*. Vol. I. *Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR 28 #158.

[M] A. A. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. 9 (1945), 3–64; English transl., Amer. Math. Soc. Transl. (1) 8 (1962), 195–272. MR 7, 7.

[TSR] T. Soundararajan, *Weakly Hausdorff spaces and the cardinality of topological spaces*, General Topology and its Relations to Modern Analysis and Algebra, III (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, pp. 301–306. MR 44 #976.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA, CANADA

DEPARTMENT OF MATHEMATICS, MEMPHIS STATE UNIVERSITY, MEMPHIS, TENNESSEE 38152

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COSTA RICA, SAN JOSE, COSTA RICA