CENTRAL APPROXIMATE UNITS IN A CERTAIN IDEAL OF $L^1(G)$

ERNST KOTZMANN AND HARALD RINDLER

ABSTRACT. In this paper we show that for a locally compact group G the ideal $L^0(G) = \{f|f \in L^1(G), \int f = 0\}$ of $L^1(G)$ has multiple approximate units belonging to the center of $L^0(G)$ iff G has a basis of invariant neighbourhoods of 1 and if all conjugacy classes of G are precompact, or, equivalently, the group of inner automorphisms is precompact in the group of all topological automorphisms. In a sense this is part of the problem to characterize certain classes of groups by properties of the group algebra.

Analogously to the papers of Reiter [7] and Mosak [5] we will characterize the class $[FIA]^-$ by the existence of approximate units in a certain ideal of the group algebra. The class $[FIA]^-$ consists of all groups such that the group of inner automorphisms is precompact in the group of all continuous automorphisms (see [1, p. 2]). It is known that $[FIA]^- = [FC]^- \cap [SIN]$, where $[SIN]$ is the class of all groups having a basis of neighbourhoods of 1 invariant under all inner automorphisms, and $[FC]^-$ is the class consisting of all groups such that every conjugacy class is precompact (see [2, p. 325, Theorem 4.1]).

In [8] Reiter shows the following theorem:

(1) A locally compact group G has the property P_1 (this property is equivalent to the amenability of G) iff the ideal $L^0(G)$ consisting of all functions f such that $f \in L^1(G)$ and $\int f = 0$, has multiple, approximate, left units (i.e. there exists a net $(u_\alpha)_{\alpha \in A}$ such that $\lim_\alpha u_\alpha * f = f$ for every $f \in L^0(G)$).

In [5] Mosak characterizes the class $[SIN]$ by:

(2) $G \in [SIN]$ iff $L^1(G)$ has bounded, multiple, approximate, two-sided units belonging to the center of $L^1(G)$.

We prove the following

THEOREM. *The following statements are equivalent:*

- (i) $G \in [FIA]^-$.
- (ii) $L^0(G)$ has multiple, approximate units belonging to the center of $L^0(G)$.

REMARK. Every abelian or compact and, more generally, every central group is an $[FIA]^-$ -group.

First we prove

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PROPOSITION. *The following statements are equivalent:*

- (a) $G \in [FC]^-$.
 (b) *Given any $\varepsilon > 0$ and any compact subset $K \subset G$, there exists a central function $s \in L^1(G)$ such that $s \geq 0$, $\int s = 1$, $\|L_y s - s\|_1 < \varepsilon$, $y \in K$, where L_y denotes the operator on $L^1(G)$ defined by $L_y s(x) = s(y^{-1}x)$, for a.e. $x \in G$.*
 (c) *There exists a net $(w_\beta)_{\beta \in B}$, $\int w_\beta = 1$, such that w_β belongs to the center of $L^1(G)$ and $\lim_\beta w_\beta * f = 0$ for all $f \in L^0(G)$.*

REMARK. If we omit the property "central" in (b), we obtain just the property P_1 .

PROOF. (a) $\Rightarrow$ (b). First we consider the discrete case. It follows from [3, p. 253, Satz 1], that for every compact, invariant subset K_0 of a discrete group $G \in [FC]^-$ the formula

$$(+)\quad \lim_n m_G(K_0^{n+1})/m_G(K_0^n) = 1$$

holds, where $m_G(K_0)$ denotes the Haar measure of the set K_0 . It follows that for every $\varepsilon > 0$ there exists an integer $n \in \mathbb{N}$ such that

$$(++)\quad m_G(K_0^{n+1} \setminus K_0^n) m_G(K_0^n)^{-1} < \varepsilon$$

(we may assume that $K_0^n \subset K_0^{n+1}$). Now let K be any compact subset and $\varepsilon > 0$; there exists an invariant compact set K_0 such that $K_0 \supset K$. Set $s = m_G(K_0^n)^{-1} \chi_{K_0^n}$, where $\chi_{K_0^n}$ denotes the characteristic function of K_0^n . Then $\int s = 1$ and s belongs to the center of $L^1(G)$. By (++), a routine calculation shows that $\|L_y s - s\|_1 < \varepsilon$, $y \in K$.

In the general case we apply the structure theorem for $[FC]^-$ -groups (see [9, Theorem 3D] and [4]). Hence there exists a compact, normal subgroup H such that $G/H = V \times D$, where V is a vector group and D is a discrete $[FC]^-$ -group. Now let K be given as before. We denote the canonical map of G onto G/H by π_H . There exist compact sets $K_1 \subset V$, $K_2 \subset D$ such that $K_1 \times K_2 \supset \pi_H(K)$. The result for discrete groups combined with [7, p. 112, Chapter 5, 2.1(i)], yields the existence of a central function $\dot{s}$ with $\dot{s} \in L^1(G/H)$, $\int \dot{s} = 1$, $\dot{s} \geq 0$, such that $\|L_{\dot{y}} \dot{s} - \dot{s}\|_1 < \varepsilon$, $\dot{y} \in K_1 \times K_2$. If we put $s = \dot{s} \cdot \pi_H$, one can easily see that s has all desired properties.

(b) $\Rightarrow$ (c). Denote by B the set of all pairs (K, ε) with K compact and $\varepsilon > 0$. B is filtering with respect to the relation $<$, where $(K, \varepsilon) < (K', \varepsilon')$ means $K \subset K'$ and $\varepsilon' < \varepsilon$. Now for each $\beta \in B$ there exists an $s_{K, \varepsilon}$ such that $\|L_y s_{K, \varepsilon} - s_{K, \varepsilon}\|_1 < \varepsilon$, $y \in K$. Taking $w_\beta = s_{K, \varepsilon}$, we see by [7, p. 113, Lemma] that w_β belongs to the center of $L^1(G)$ and that every $f \in L^0(G)$ satisfies $\lim_\beta f * w_\beta = 0$.

(c) $\Rightarrow$ (a). Since the center of $L^1(G)$ is not trivial ($w_\beta \neq 0$), there exists a compact, invariant neighbourhood U of the identity (see [5]). Set $g = m_G(U)^{-1} \chi_U$; it follows that $v_\beta = w_\beta * g$ is a continuous, central function vanishing at infinity. Furthermore $\int v_\beta = 1$ and

$$(+++) \quad \lim_\beta \|f * v_\beta\|_1 = \lim_\beta \|f * w_\beta * g\|_1 \leq \lim_\beta \|f * w_\beta\|_1 = 0.$$

Let H be the normal subgroup consisting of all elements with precompact

conjugacy class. H is an open subgroup of G since $U \subset H$. It is known that $f \in L^1(G)$ is central iff f satisfies, for all $x \in G$, $f(xy x^{-1}) = f(y)\Delta(x)$ a.e. $y \in G$, where the null set may depend on x (see [6, Proposition 1.2]). Since the center of $L^1(G)$ is not trivial, G is unimodular [5]. Hence we obtain that $v_\beta(xy x^{-1}) = v_\beta(y)$ for all $x, y \in G$, because v_β is continuous, $\beta \in B$. Therefore the conjugacy class of each $y \in G$ satisfying $v_\beta(y) \neq 0$ for any $\beta \in B$ is precompact (v_β is continuous and vanishes at infinity); in particular $\text{Supp } v_\beta \subset H$. If $G \neq H$, we take an element $a \in G$ such that $aH \cap H = \emptyset$. Choosing any $f_1 \in L^1(G)$ satisfying $f_1 \geq 0$, $\int f_1 = 1$, $\text{Supp } f_1 \subset H$, we put $f = L_a f - f$ and obtain

$$\text{Supp}(v_\beta * L_a f_1) \cap \text{Supp}(v_\beta * f_1) \subset H \cdot aH \cap H \cdot H = \emptyset.$$

Hence we get

$$\|f * v_\beta\|_1 = \|v_\beta * L_a f_1\|_1 + \|v_\beta * f_1\|_1 \geq \left(\int v_\beta\right) \cdot \left(\int f_1\right) = 1$$

in contradiction to (+++). Therefore $G = H$, i.e. $G \in [FC]^-$.

Now we are able to prove the theorem.

(i) $\Rightarrow$ (ii). (2) yields the existence of bounded, multiple, approximate, two-sided units $(v_\alpha)_{\alpha \in A}$ belonging to the center of $L^1(G)$ such that $\int v_\alpha = 1$, $\alpha \in A$. Since $G \in [FC]^-$, it follows by the proposition above that there exists a net $(w_\beta)_{\beta \in B}$ in the center of $L^1(G)$ with $\int w_\beta = 1$ and $\lim_\beta w_\beta * f = 0$, $f \in L^0(G)$. If we put $u_{\alpha,\beta} = v_\alpha - w_\beta$, one can easily see that $u_{\alpha,\beta}$ belongs to the center of $L^0(G)$, and, moreover, every $f \in L^0(G)$ satisfies

$$\lim_{\alpha,\beta} f * u_{\alpha,\beta} = \lim_\alpha f * v_\alpha - \lim_\beta f * w_\beta = f,$$

i.e. $(u_{\alpha,\beta})_{\alpha \in A, \beta \in B}$ is the desired approximate unit.

(ii) $\Rightarrow$ (i). First we show that $G \in [SIN]$. Let f belong to the center of $L^0(G)$. We want to show that $f * g = g * f$ holds for every $g \in L^1(G)$, i.e. f is in the center of $L^1(G)$. We assume that $\int g = 1$. Let $(v_\beta)_{\beta \in B}$ be a bounded, multiple, approximate, two-sided unit in $L^1(G)$ with $\int v_\beta = 1$, $\beta \in B$. Hence we obtain

$$\begin{aligned} f * g - g * f &= \lim_\beta f * (g - v_\beta) - \lim_\beta (g - v_\beta) * f \\ &\quad + \lim_\beta f * v_\beta - \lim_\beta v_\beta * f \\ &= 0, \end{aligned}$$

because $g - v_\beta$ belongs to $L^0(G)$, $\beta \in B$. This fact implies that $\{u_\alpha\}$ is contained in the center of $L^1(G)$. Next we observe that the operator on $L^1(G)$ of left translation by x cannot reduce to the identity on $\{u_\alpha\}$ (otherwise $L_x f = f$ holds for any $f \in L^0(G)$; a trivial modification of the proof in [5, p. 615] shows that this is a contradiction). Now one can imitate step by step the proof of [5]. Therefore $G \in [SIN]$. To show that $G \in [FC]^-$ it suffices, by the proposition, to show that there exists a net $(w_\gamma)_{\gamma \in C}$ belonging to the center of $L^1(G)$ such that $\int w_\gamma = 1$ and $\lim_\gamma w_\gamma * f = 0$, $f \in L^0(G)$. If we put $w_{\alpha,\beta} = v_\beta - u_\alpha$, then $w_{\alpha,\beta}$ belongs to the center of $L^1(G)$; furthermore $\int w_{\alpha,\beta} = 1$

and $\lim_{\alpha, \beta} w_{\alpha, \beta} * f = 0, f \in L^0(G)$. Hence $G \in [FC]^- \cap [SIN]$, i.e. $G \in [FIA]^-$.

The methods of our proof admit the following

COROLLARY. *If $L^0(G)$ possesses central approximate units, then $L^0(G)$ possesses even bounded, multiple, approximate units consisting of continuous functions with compact support, the norm of the units being bounded by the constant 2.*

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MATHEMATICS INSTITUTE, UNIVERSITY OF VIENNA, A-1090 VIENNA, AUSTRIA