ERGODIC THEOREMS OF WEAK MIXING TYPE

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ABSTRACT. Given a linear contraction T on a Banach space X and $x \in X$, the convergence

$$\forall x^* \in X^* N^{-1} \sum_{i=1}^N |\langle x^*, T^i x \rangle| \xrightarrow{N \rightarrow \infty} 0$$

is shown to be equivalent to the convergence

$$\sup_{\|x^*\| \leq 1} N^{-1} \sum_{j=1}^N |\langle x^*, T^{k_j} x \rangle| \rightarrow 0$$

for every subsequence with k_j/j bounded. A sufficient condition is that, for some $\{n_i\}$, $T^{n_i} x \rightarrow 0$ weakly.

The following theorem gives a characterization of weak mixing for operators in general Banach spaces, generalizing the well-known results for measure preserving transformations. For more details see [2].

THEOREM. Let T be a linear operator on a Banach space with $\sup_{n \geq 0} \|T^n\| < \infty$, and let $x \in X$. Then the following conditions are equivalent.

(i) For every $x^* \in X^*$ we have $\lim_{N \rightarrow \infty} N^{-1} \sum_{j=1}^N |\langle x^*, T^j x \rangle| = 0$.

(ii)
$$\sup_{\|x^*\| \leq 1} N^{-1} \sum_{j=1}^N |\langle x^*, T^j x \rangle| \xrightarrow{N \rightarrow \infty} 0.$$

(iii) For every subsequence $\{k_j\}$ with positive lower density (i.e., $\sup k_j/j < \infty$) we have $\lim_{N \rightarrow \infty} \|N^{-1} \sum_{j=1}^N T^{k_j} x\| = 0$.

(iv) For every subsequence $\{k_j\}$ with positive lower density,

$$\sup_{\|x^*\| \leq 1} N^{-1} \sum_{j=1}^N |\langle x^*, T^{k_j} x \rangle| \xrightarrow{N \rightarrow \infty} 0.$$

PROOF. (i) $\Rightarrow$ (ii): We first assume $\|T\| \leq 1$. Let $B = \{x^* \in X^*: \|x^*\| \leq 1\}$. B is a compact Hausdorff space in the w^* -topology, and $T^*(B) \subset B$. Since T^* is w^* -continuous on B , the operator $Af(x^*) = f(T^*x^*)$ is a contraction of $C(B)$. For $g(x^*) = |\langle x^*, x \rangle|$, we have by (i) that $N^{-1} \sum_{j=1}^N A^j g \rightarrow 0$ pointwise. Hence for $\mu \in C(B)^*$ satisfying $A^* \mu = \mu$ we have $\langle \mu, g \rangle = 0$,

Received by the editors March 21, 1975.

AMS (MOS) subject classifications (1970). Primary 47A35; Secondary 28A65, 60J05.

Key words and phrases. Ergodic theorems, weak mixing.

¹ Research supported in part by NSF grant GP 34118.

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and $g \in \overline{(I - A)C(B)}$, yielding $\|N^{-1} \sum_{j=1}^N A^j g\| \rightarrow 0$ and (ii) holds. Now let $\sup_{n \geq 0} \|T^n\| = M$, $1 < M < \infty$. We define an equivalent norm in X by $\|y\| = \sup \|T^n y\|$. Then $\|T\| \leq 1$, and since (i) is not altered, we have, by the previous proof and using $\|x^*\| \leq \|x^*\|$, that

$$\sup_{\|x^*\| \leq 1} N^{-1} \sum_{j=1}^N |\langle x^*, T^j x \rangle| \leq \sup_{\|x^*\| \leq 1} N^{-1} \sum_{j=1}^N |\langle x^*, T^j x \rangle| \xrightarrow{N \rightarrow \infty} 0.$$

(ii) $\Rightarrow$ (iv): Let $k_j/j \leq a < \infty$. Then

$$\begin{aligned} \sup_{\|x^*\| \leq 1} N^{-1} \sum_{j=1}^N |\langle x^*, T^{k_j} x \rangle| &\leq \sup_{\|x^*\| \leq 1} N^{-1} \sum_{i=1}^{k_N} |\langle x^*, T^i x \rangle| \\ &\leq a \sup_{\|x^*\| \leq 1} (k_N)^{-1} \sum_{i=1}^{k_N} |\langle x^*, T^i x \rangle| \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

(iv) $\Rightarrow$ (iii) is immediate.

(iii) $\Rightarrow$ (i) is proved (easily) in the last paragraph of [2, Theorem IV].

PROPOSITION. *Let $\sup \|T^n\| < \infty$ and $x \in X$. If for some subsequence $\{n_i\}$, $T^{n_i} x \rightarrow 0$ weakly, then (ii) holds.*

PROOF. When $\|T\| \leq 1$, let A on $C(B)$ and g be as in the proof of the Theorem. Then $A^{n_i} g \rightarrow 0$ pointwise, and $\langle \mu, g \rangle = 0$ for every $\mu \in C(B)^*$ satisfying $A^* \mu = \mu$. The remainder of the proof is exactly the same as before.

REMARKS. 1. If X^* is separable, then any of the conditions of the Theorem implies the existence of a subsequence $\{n_i\}$ such that $T^{n_i} x \rightarrow 0$ (by the diagonal process, we obtain $\{n_i\}$ such that $\langle x_k^*, T^{n_i} x \rangle \rightarrow 0$ on a dense $\{x_k^*\}$), and $\{n_i\}$ even has density 1.

2. The Proposition and the Theorem together improve the result of [2], where the assumption was $T^{n_i} x \rightarrow 0$ for some $\{n_i\}$ of density one (see following example). They also eliminate the use of the combinatorial machinery of [2].

3. The Proposition can also be proved by using the Banach-Mazur theorem, as in [3]. The simpler argument was suggested by Professor H. Furstenberg.

LEMMA. *Let T be a positive contraction of L_1 , having no fixed points (i.e., $Tf = f \Rightarrow f = 0$). If $g \in L_1$ and $\{n_i\}$ is an increasing sequence of density one such that $T^{n_i} g \rightarrow 0$ weakly, then $\|T^n g\|_1 \rightarrow 0$.*

PROOF. Let $\{a_i\}$ be a bounded sequence of numbers, with $|a_i| \leq a$. If $\{n_i\}$ has density one (i.e., $n_N/N \rightarrow 1$), then

$$\begin{aligned} &\left| N^{-1} \sum_{i=1}^N (a_i - a_{n_i}) \right| \\ &= \left| N^{-1} \sum_{i=1}^N a_i - N^{-1} \sum_{i=1}^{n_N} a_i + N^{-1} \sum \{a_j : j \notin \{n_i\}, j < n_N\} \right| \\ &\leq 2(n_N - N)a/N \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

Hence, if L is a Banach limit which preserves mean convergence [1, p. 33], $L\{a_i\} = L\{a_{n_i}\}$. The Lemma follows from the proof of Theorem 5.1 of Krengel and Sucheston [5], using that particular Banach limit.

EXAMPLE. Let $\mathcal{T}$ be an *invertible* weakly mixing transformation with no finite invariant measure, and T the corresponding L_1 contraction (see Kakutani and Parry [4]). If $\int g = 0$, then the conditions of our theorem are satisfied, and the Lemma shows that $T^{n_i}g$ does not converge weakly to 0 for $\{n_i\}$ of density one (because of $\|T^{-1}\| = 1$).

PROBLEM. Is the condition in the Proposition necessary for the conditions of the Theorem? Since the conditions of the Theorem imply that 0 is a weak cluster point of $\{T^n x\}$, it is necessary if $\{T^n x\}$ is weakly sequentially compact, and it is also necessary when X^* is separable.

In [6], R. Nagel studied a similar problem in the case that for *every* $x \in X$, $\{T^n x\}$ is weakly sequentially compact (see also [3]).

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