SMALL SETS OF HOMEOMORPHISMS WHICH CONTROL MANIFOLDS

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ABSTRACT. Let M^n be a smooth connected paracompact manifold without boundary. A set D of self-homeomorphisms of M^n to itself is called controllable iff the semigroup generated by D acts transitively on M^n .

THEOREM A. *There is a complete vector field X on M^n and a self-homeomorphism H so that the set D consisting of H , H^{-1} and X_t , $t \in \mathbb{R}$, is controllable.*

THEOREM B. *Let $n \neq 4$ and let M^n be compact and orientable. If n is even, > 6 , let M^n be simply connected. If $n \equiv 0 \pmod{4}$, let signature $M^n = 0$. Then there is a vector field X on M^n and a self-homeomorphism H so that the set consisting of H , H^{-1} and X_t , $t > 0$, is controllable.*

The purpose of this paper is to extend and modify some results of the author and H. Sussmann [5] concerning the controllability of a manifold by means of two vector fields. The reader is advised that [5] is an indispensable reference for the material below.

Let M^n be a smooth, connected, paracompact manifold without boundary. Consider a set D of homeomorphisms from M^n to itself. For our purposes, we shall call D *controllable* if and only if the semigroup of homeomorphisms generated by D acts transitively on M^n . That is to say, D is controllable provided that for any points $x, y \in M^n$ there is a homeomorphism Φ of M^n to itself with $\Phi(x) = y$ and Φ of the form $\Phi = \phi_1^{n_1} \cdots \phi_j^{n_j}$ where $n_i \geq 0$ and $\phi_i \in D$.

In the context of [5], controllability was discussed for sets of smooth complete vector fields $V^1 \cdots V^k$. Let V_t^i be the diffeomorphism which is obtained by applying the flow generated by V^i for time t . Then, in the current terminology $V^1, \dots, V^k$ is controllable if and only if the set $\{V_t^i, i = 1, 2, \dots, k, t \geq 0\}$ is controllable. The main result of [5] is that, given M^n , one can always find a pair of controllable vector fields X, Y , that is, the set $\{X_t, t \geq 0\} \cup \{Y_t, t \geq 0\}$ will be controllable.

The aim of the present paper is to find a controllable set D which is, in some sense, even "smaller" than that found in [5].

We state our results:

THEOREM A. *Let M^n be a connected paracompact smooth manifold without boundary. Then there is a smooth, complete vector field X on M^n and a homeomorphism $H: M^n \rightarrow M^n$ so that the set D consisting of H , H^{-1} and all*

Received by the editors September 29, 1975.

AMS (MOS) subject classifications (1970). Primary 57D25; Secondary 57D70.

Key words and phrases. Controllable set of vector fields, controllable set of homeomorphisms, gradient-like vector field, topologically conjugate vector fields, twisted double theorem.

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diffeomorphisms of the form X_t , $t \in \mathbf{R}$, is a controllable set.

In other words, Theorem A says that in the group of self-homeomorphisms of M^n , the subgroup generated by D (which is the same as the semigroup generated by D , since D is closed under taking of inverses) acts transitively on M^n . In fact, we actually obtain a slightly finer result.

THEOREM B. *Let M^n be a compact, connected orientable smooth manifold without boundary, $n \neq 4$. If $n \equiv 0 \pmod{4}$, let the signature of M^n be zero. If $n \geq 5$ let M^n be simply connected.*

Then there is a vector field X on M^n and a homeomorphism $H: M^n \rightarrow M^n$ so that the set D consisting of H , H^{-1} and all homeomorphisms of the form X_t , $t \geq 0$, is a controllable set.

(Recall that if M^n is a closed oriented manifold of dimension $n \equiv 0 \pmod{4}$, the signature is an invariant of oriented homotopy type defined as follows:

Since $n/2$ is an even integer, the cohomology cup-product defines a symmetric bilinear pairing $H^{n/2}(M^n; \mathbf{R}) \otimes H^{n/2}(M^n; \mathbf{R}) \rightarrow \mathbf{R}$. The signature of this quadratic form is, by definition, the signature of the manifold. (See [3] for details.))

The proofs of Theorems A and B require a careful analysis of the proof of the main result of [5]. We therefore give a short outline of that proof. Recall that the purpose of [5] is to find a pair of controllable vector fields.

We start off in [5] with two vector fields having the following properties. X^0 is a gradient-like vector field for some proper Morse function f on M^n having a single relative and absolute minimum. (See [6], [7] for a good account of Morse functions.) This means $X^0(f) \geq 0$ and is equal to 0 only at the critical points of f where X^0 also vanishes. Y^0 is the negative of X^0 . In point of fact, however, all the techniques of [5] will go through if Y^0 is gradient-like for some Morse function having the same critical points as f and if $Y^0 = -X^0$ in a neighborhood of every critical point.

We then find a certain diffeomorphism $G: M^n \rightarrow M^n$ and let

$$Y^1(x) = G_*(Y^0(G^{-1}(x)))$$

for $x \in M^n$. G is a perturbation which is the identity except near the critical points. The effect of this change is simply to shift the critical points of Y^1 away from those of X^0 and to put "stable" and "unstable" manifolds for both flows in general position near critical points.

We emerge with a certain picture of the behavior of X^0 and Y^1 near the original minimum point p of f . In particular, there is a disc D^n surrounding p with local coordinates $(x^1, \dots, x^n)$. In these coordinates $X^0(x^1, \dots, x^n) = x^1(\partial/\partial x^1) + \dots + x^n(\partial/\partial x^n)$, i.e., X is the "outward pointing" vector field.

$$f(x^1 \cdots x^n) = (x^1)^2 + \dots + (x^n)^2.$$

Y^1 is $-X^0$ near the boundary S^{n-1} of D^n , but the flow lines converge to a single "sink" q different from p .

We then construct a certain pair of vector fields, A and B , which vanish outside a small collar neighborhood of S^{n-1} in D^n . We set $X = X^0 + A$,

$Y = Y^1 + B$. It then turns out that X, Y is the desired controllable pair.

(For the reader who may wish to check the details of this construction in [5], it should be pointed out that, in the outline above, the notation is somewhat different from that used in [5]. The vector field X^0 above is also denoted X^0 in [5], but our Y^1 is the Y^0 of [5]. A and B are, respectively, denoted by $\phi(\xi)\bar{X}(p)$ and $\phi(\xi)\bar{Y}(p)$ in [5, §7].)

We are now ready to prove Theorem A. Referring to the outline above, we assume that X and Y have been constructed starting from X^0 and $Y^0 = -X^0$.

DEFINITION. Two vector fields V and W are said to be topologically (resp., smoothly) conjugate if and only if there is a homeomorphism (resp. diffeomorphism) H so that $H \cdot V_t = W_t \cdot H$ for all $t \in \mathbf{R}$. H is called the conjugating homeomorphism (resp. diffeomorphism).

Note that in the outline above, Y^1 is constructed so as to be smoothly conjugate to Y^0 with G the conjugating diffeomorphism.

Furthermore, we claim that X , as constructed above, is topologically conjugate to X^0 and Y is topologically conjugate to Y^1 . The proof of this fact is as follows: In constructing X, Y , it turns out that in the disc, D^n , each trajectory of X (except the degenerate trajectory which is the single point p) flows outward from p and reaches the boundary S^{n-1} . (See [5, §7].) That is, adding A to X^0 to obtain X introduces no new zeroes of the vector field and we still have $X(f) \geq 0$. (In fact, A is constructed so that $X(f) = X^0(f)$.) Now let G be defined as follows: Set $G = \text{identity on the closure of } M^n - D^n$. Let $G(p) = p$. Any point in $D^n - p$ is of the form $X_t^0(x)$ for some unique $s \in S^{n-1}$, $t \leq 0$. Similarly any point in $D^n - p$ is also of the form $X_t(s)$ in a unique way. We let $G(X_t^0(s)) = X_t(s)$.

Since this map is the identity on S^{n-1} we obtain a well-defined map $G: M^n \rightarrow M^n$ which is, in fact, a homeomorphism throwing the flow lines of X^0 onto the flow lines of X . (To check that G and its inverse are continuous at p , note that G sends $f^{-1}(\varepsilon)$ homeomorphically onto itself for $f(p) \leq \varepsilon \leq f(S^{n-1})$.) G is thus the conjugating homeomorphism.

Similarly, Y is conjugate to Y^1 by an identical construction (although checking continuity at q is a bit less direct).

Now note that topological conjugacy is implied by smooth conjugacy and is an equivalence relation; moreover, if V is topologically conjugate to W , then $-V$ is topologically conjugate to $-W$. Thus, X is topologically conjugate to $-Y$ since in the sequence $X, X^0, -Y^0, -Y^1, -Y$ each element is topologically conjugate to the next (recall we assume $Y^0 = -X^0$).

Let H be the conjugating homeomorphism, i.e.,

$$(1) \quad H \cdot X_t = -Y_t \cdot H$$

or equivalently

$$(1') \quad H \cdot X_t \cdot H^{-1} = Y_{-t}.$$

The main result of [5] now assures us that, given $x, y \in M^n$, we may write

$$y = X_{t_1} Y_{s_1} Y_{t_2} Y_{s_2} \cdots X_{t_n} Y_{s_n}(X)$$

for some $t_i, s_i \geq 0$.

Applying (1') we obtain

$$y = Y_{t_1} H X_{-s_1} H^{-1} X_{t_2} \cdots X_{t_n} H X_{-s_n} H^{-1} (X).$$

Thus, X and H play the role demanded in Theorem A and the proof of Theorem A is complete.

REMARK. Note that in order to move x to y we need only apply a homeomorphism which is a concatenation of X_i 's, H and H^{-1} , where between any two X_i 's there occurs only H or H^{-1} and not any higher powers.

To prove Theorem B, we must use a few deeper results from differential topology. Assume, henceforth, that M^n satisfies the hypothesis of Theorem B, i.e., M^n compact, connected, without boundary and orientable, $n \neq 4$, and if n even, ≥ 6 , M^n simply connected, and signature $M^n = 0$ if $n \equiv 0 \pmod{4}$. We recall the "twisted double" theorem, which is known classically for 2 and 3 dimensional manifolds and which has been established for manifolds of dimension 5 and greater, under suitable restrictions, by a series of successively sharper results [10], [4], [11].

We quote the sharpest of these:

PROPOSITION (WINKELNKEMPER [11]).¹ Let M^n satisfy the conditions above. Then M^n may be decomposed as $M^n = W_0^n \cup W_1^n$ (where W_0^n, W_1^n are connected codimensional-0 submanifolds with common boundary $\partial = \partial W_0^n = \partial W_1^n$) such that M^n admits a self-diffeomorphism $G: M^n \rightarrow M^n$ with $G(W_0^n) = W_1^n, G(W_1^n) = W_0^n$.

We remark that for some bicollar neighborhood $\partial \times [-1, 1]$ of ∂ , $G: \partial \times [-1, 1] \rightarrow \partial \times [-1, 1]$ and is given by $G(x, t) = (G(x), -t)$ for $x \in \partial$, $t \in [-1, 1]$.

Since W_0^n is connected we may find a Morse function $f: W_0^n \rightarrow [0, a]$ with a single relative and absolute minimum p and with $\partial = f^{-1}(a)$ having no critical points. We extend f to all of M^n by setting $f(x) = 2a - f(G(x))$ for $x \in W_1^n$.

If c is a critical point of f recall (cf. [6], [7]) that there are "nice" local coordinates $(x_c^1 \cdots x_c^i, y_c^1 \cdots y_c^{n-i})$ for a neighborhood around c so that c has coordinates $(0, \dots, 0)$ and so that

$$\begin{aligned} f(x_c^1 \cdots x_c^i, y_c^1 \cdots y_c^{n-i}) \\ = f(c) - (x_c^1)^2 - (x_c^2)^2 - \cdots - (x_c^i)^2 + (y_c^1)^2 + \cdots + (y_c^{n-i})^2 \end{aligned}$$

(i is the index of the critical point).

Clearly, if $c \in W_0^n$ is a critical point of f , so is $d = G(c) \in W_1^n$ with index $c = n - \text{index } d$. Moreover, we may obviously choose nice local coordinates for c and d , respectively, so that

$$x_d^r(G(x)) = y_c^r(x), \quad y_d^r(G(x)) = x_c^r(x)$$

for x in the coordinate patch around c .

We now modify G to produce a new diffeomorphism $\bar{H}: M^n \rightarrow M^n$ which also interchanges W_0^n and W_1^n . For each critical point $d \in W_1^n$, $d = G(c)$;

¹We should point out that Winkelkemper's result is stated for simply connected M^n , $n > 6$. Barden's thesis [1] allows Winkelkemper's method to be extended to the case where n is odd, > 5 , M^n non-simply-connected.

thus critical points in W_1^n correspond 1-1 to critical points in W_0^n . Write this correspondence as $d = d(c)$, $c = c(d)$. Now G will *not*, in general, throw nice coordinates around the critical point $d \in W_1^n$ onto nice coordinates around $c(d)$. In fact $G(c)$ is not necessarily $d(c)$. But we may apply the lemma of Cerf [2] and Palais [8] in order to isotop $G|W_1^n$ to a diffeomorphism $G_1: W_1^n \rightarrow W_0^n$ so that:

- (a) $G_1 = G|W_1^n$ in a neighborhood of ∂ ,
 - (b) $G_1(d) = c(d)$ for all critical points d of f in W_1^n .
 - (c) For respective nice coordinates around d and $c = c(d)$,
- $$x'_d(G_1(x)) = y'_d(x); \quad y'_c(G_1(x)) = x'_d(x).$$

We let

$$\bar{H} = \begin{cases} G & \text{on } W_0^n, \\ G_1 & \text{on } W_1^n. \end{cases}$$

$\bar{H}$ has properties:

(α) $\bar{H}$ permutes the critical points of f taking critical points c of index i to critical points $d = d(c)$ of index $n - i$.

(β) For nice coordinates around c and d ,

$$x'_d(\bar{H}(x)) = y'_c(x); \quad y'_c(\bar{H}(x)) = x'_d(x).$$

Note that (α) and (β) also hold when $\bar{H}$ is replaced by its inverse $\bar{H}^{-1}$.

Pick a Riemann metric for M^n so that in a neighborhood of each critical point c of f with local coordinates $x_c^1 \cdots x_c^i, y_c^1 \cdots y_c^{n-i}$, the metric is the standard one with orthonormal basis $\partial/\partial x_c^1 \cdots \partial/\partial x_c^i, \partial/\partial y_c^1 \cdots \partial/\partial y_c^{n-i}$.

Let $X^0 = \frac{1}{2} \cdot \text{grad } f$ (with respect to this metric). Set $g = f \cdot \bar{H}^{-1}$ and let Y^0 be the vector field given by

$$Y^0(x) = \bar{H}_*(X^0(\bar{H}^{-1}(x))).$$

Clearly, Y^0 is smoothly conjugate to X^0 . We also have

$$\begin{aligned} Y^0(x)(g) &= [\bar{H}_* X^0(\bar{H}^{-1}(x))] [f \cdot \bar{H}^{-1}] \\ &= [X^0(\bar{H}^{-1}(x))] [f \cdot \bar{H}^{-1} \cdot \bar{H}] = [X^0(\bar{H}^{-1}(x))](f). \end{aligned}$$

Thus, $Y^0(x)(g) \geq 0$ for all x and is 0 only if $\bar{H}^{-1}(x)$ is a critical point of f , i.e., x is a critical point of g . Therefore Y^0 is a gradient-like vector field for the Morse function g . Now it is clear that critical points of g of index i are critical points of f of index $n - i$ and that $Y^0 = -X^0$ in a neighborhood of every critical point.

We may now apply the argument of [5], as outlined above, to produce a controllable pair of vector fields X, Y with X topologically conjugate to X^0 and Y topologically conjugate to Y^0 . Since topological conjugacy is an equivalence relation, X is topologically conjugate to Y by a homeomorphism H so that $H \cdot X_t \cdot H^{-1} = Y_t$ for all t . For each $x, y \in M^n$ we have

$$y = X_{t_1} Y_{s_1} \cdots X_{t_n} Y_{s_n}(x), \quad t_i, s_i \geq 0,$$

which we rewrite

$$y = X_{t_1} H X_{s_1} H^{-1} \cdots X_{t_n} H X_{s_n} H^{-1}(x).$$

Thus, X , H play the role demanded by Theorem B and the proof of Theorem B is complete.

REMARK. Note that to move x to y we need only apply a concatenation of X_t 's, $t \geq 0$, H , and H^{-1} where between any two H_t 's there occurs only H or H^{-1} , but no higher powers.

REFERENCES

1. D. Barden, *The structure of manifolds*, Thesis, Cambridge, 1963.
2. J. Cerf, *Topologie de certains espaces plongements*, Bull. Soc. Math. France **89** (1961), 227–380. MR **25** #3543.
3. F. Hirzebruch, *Topological methods in algebraic geometry*, 3rd ed., Die Grundlehren der math. Wissenschaften, Band 131, Springer-Verlag, New York, 1966. MR **34** #2573.
4. N. Levitt, *Applications of engulfing*, Thesis, Princeton University, 1967.
5. N. Levitt and H. Sussmann, *On controllability by means of two vector fields*, SIAM J. Control **13** (1975), 1271–1281.
6. J. W. Milnor, *Morse theory*, Ann. of Math. Studies, no. 51, Princeton Univ. Press, Princeton, N.J., 1963. MR **29** #634.
7. ———, *Lectures on the h-cobordism theorem*, Princeton Univ. Press, Princeton, N.J., 1965. MR **32** #8352.
8. R. Palais, *Extending diffeomorphisms*, Proc. Amer. Math. Soc. **11** (1960), 274–277. MR **22** #8515.
9. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, Leipzig, 1934; reprint, Chelsea, New York, 1947.
10. S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399. MR **27** #2991.
11. H. E. Winkelnkemper, Thesis, Princeton University, 1970.

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