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A CLASS OF SPECTRAL SETS

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ABSTRACT. The two main results are:

(i) If the union and intersection of two closed sets are Ditkin sets, then each of the sets is a Ditkin set.

(ii) If the union of two sets is a spectral set and their intersection is a Ditkin set, then each of the sets is a spectral set.

A corollary of (i) is a generalization of a theorem due to Calderón which proved that closed polyhedral sets in R^n are Ditkin (= Calderón) sets. A corollary of (ii) establishes an analogous result for spectral sets.

The proofs hold for commutative semisimple regular Banach algebras which satisfy Ditkin's condition–that the empty set and singletons are Ditkin sets in the maximal ideal space.

Let G be the character group of the locally compact abelian group Γ, and let $A(G)$ denote the algebra of all Fourier transforms of functions φ in $L^1(\Gamma)$ with the pointwise product. Thus f belongs to $A(G)$ if and only if $f(s) = \hat{\varphi}(s) = \int_\Gamma \varphi(\gamma)\overline{(s,\gamma)}\,d\gamma$ for some φ in $L^1(\Gamma)$. If $A(G)$ is equipped with the $L^1(\Gamma)$ norm, i.e. $\|f\| = \int_\Gamma |\varphi(\gamma)|\,d\gamma$, then $A(G)$ becomes a commutative Banach algebra isometrically isomorphic to $L^1(\Gamma)$.

If I is an ideal in $A(G)$, then $Z(I)$, the zero set of I, is the intersection of the zero sets of all the elements f belonging to I, i.e. $Z(I) = \bigcap\{Z(f): f \in I\}$. Let E be a closed subset of G. The largest ideal (hence closed) in $A(G)$ with zero set E is denoted by $k(E)$, and the smallest ideal with zero set E is denoted by $j(E)$. We note that

$$k(E) = \{f \in A(G): f = 0 \text{ on } E\}$$

and

$$j(E) = \{f \in A(G): f \text{ has compact support disjoint from } E\}.$$

If E is a closed subset of G, then E is a spectral set (a set for which spectral synthesis holds) if the $A(G)$-closure of $j(E) = \overline{j(E)}$ is the only closed ideal with zero set E [7, p. 158]. The closed set E is said to be a Ditkin set or a Calderón set [7, p. 169] if, for each f in $k(E)$, f belongs to the closed ideal $\overline{f \cdot j(E)}$. (See also [4, p. 183], [2, p. 227], [5, p. 30], and [3, pp. 513–515].)

The family of spectral sets referred to in the title of this paper is the class of Ditkin sets. It is clear from the definition that every Ditkin set is a spectral set, but it is not known whether or not the two classes coincide. Ditkin sets occur frequently in the study of spectral sets, however. For example, the abstract

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Wiener Tauberian theorem follows from the fact that the empty set is a Ditkin set (hence is an S-set) and the Šilov-Wiener Tauberian theorem can be proved by showing (as in [9]) that closed scattered sets are Ditkin sets. In fact (by [9]), if a set is found to be a spectral set by means of the (generalization of the) Šilov-Wiener Tauberian theorem, then it is a Ditkin set.

1. THEOREM. *Let E and F be closed subsets of G for which $E \cap F$ is a Ditkin set. Then $E \cup F$ is a Ditkin set if and only if both E and F are Ditkin sets.*

PROOF. Suppose that $E \cup F$ is a Ditkin set and that $f \in k(E)$.

Since $E \cap F$ is a Ditkin set, there is a sequence $\{u_n\}$ in $j(E \cap F)$ such that $\lim_n \|u_n f - f\| = 0$. For a fixed n, let $C = F \cap$ (support of $u_n f$). Thus C is compact and disjoint from E. Hence there is a $w \in j(E)$ such that $w = 1$ on the compact set C. Let $g = u_n f - u_n wf$, so g belongs to $\overline{g \cdot j(E \cup F)}$. Thus g belongs to $\overline{f \cdot j(E)}$; so $g + u_n fw \in \overline{f \cdot j(E)}$. Consequently, $f \in \overline{f \cdot j(E)}$. Therefore E is a Ditkin set. Similarly, F is a Ditkin set.

The other half of the theorem asserts that finite unions of Ditkin sets are Ditkin sets [7, p. 170].

2. COROLLARY. *Let A and B be closed subsets of G. If B is a Ditkin set containing A, and the relative boundary of A as a subset of B is a Ditkin set in G, then A is a Ditkin set.*

PROOF. Let $E = A$ and $F = \overline{B \backslash A}$ in Theorem 1.

This result has its origin in Calderón's Theorem [1, Theorem 5]. Calderón proved it for the case where B is a closed subgroup of G. The corollary is a generalization of his theorem since closed subgroups are Ditkin sets [1, Theorem 2]. (Also, see [7, p. 170] and [5, pp. 152–153].) The method of Theorem 1 can be employed to prove the following result.

THEOREM 1′. *If $E \cup F$ is a Ditkin set, and $F \cap$* bdry *E is a Ditkin set, then E is a Ditkin set.*

3. LEMMA. *Let E_1 and E_2 be closed subsets of G whose intersection is a Ditkin set, and let $E = E_1 \cup E_2$. Let I be a closed ideal in $A(G)$ whose zero set is E. Then I can be uniquely expressed in the form $I = I_1 \cap I_2$, where $I_i = \overline{I + J_i}$, and $J_i = j(E_i)$ $(i = 1,2)$. This form is unique in the sense that if $Z(I'_1) = E_1$, $Z(I'_2) = E_2$, and $I = I'_1 \cap I'_2$, then $I'_1 = I_1$ and $I'_2 = I_2$.*

PROOF. To prove the equality, we have only to show that $I_1 \cap I_2 \subseteq I$. Hence, let f belong to $I_1 \cap I_2$ and let $\delta > 0$ be arbitrarily chosen. Since $E_1 \cap E_2 = F$ is a Ditkin set there is a v in $j(F)$ such that $\|vf - f\| < \delta$.

The function vf belongs to I locally at the point at infinity of G, and locally at each point of G not in $E \cap$ (supp vf). Let $C_1 = E_1 \cap$ (supp vf), $C_2 = E_2 \cap$ (supp vf), and observe that $C_1 \cap E_2 = \varnothing$. Hence there is a w in $j(E_2)$ such that $w = 1$ in a neighborhood of C_1. Since $vf \in I_1$, given $\varepsilon > 0$, there is a g in I and a u in $j(E_1)$ for which $\|vfw - uw - gw\| < \varepsilon$. The ideal I is closed, and $uw \in I$ so this means that $vfw \in I$. Hence vf belongs to I locally at each point of C_1, and similarly, it belongs to I locally at each point of C_2. Consequently $vf \in I$. The choice of $\delta > 0$ is arbitrary, so it follows that $f \in I$.

To prove the uniqueness, suppose that $Z(I'_1) = E_1$, $Z(I'_2) = E_2$, and that $I_1 \cap I_2 = I'_1 \cap I'_2$. Let $f \in I'_1$, and $\varepsilon > 0$ be given. Since $f \in k(F)$ there is a u in $j(F)$ such that $\|uf - f\| < \varepsilon$. Let $K = E_1 \cap (\operatorname{supp} uf)$, and let w be chosen in $j(E_2)$ such that $w = 1$ on a neighborhood of K. Thus $uf - ufw \in I_1$. But $ufw \in j(E_2) \subset I'_2$, so $ufw \in I_1 \cap I_2$, i.e. $ufw \in I_1$. Since $uf - ufw$ belongs to I_1 also, it follows that $uf \in I_1$, hence that $f \in I_1$. Thus $I'_1 \subseteq I_1$. Similarly, $I_1 \subseteq I'_1$, so that $I'_1 = I_1$. By an identical argument it follows that $I'_2 = I_2$.

4. THEOREM. *Let E_1 and E_2 be closed subsets of G whose intersection is a Ditkin set, and let $E = E_1 \cup E_2$. Then E is a spectral set if and only if both E_1 and E_2 are spectral sets.*

PROOF. If E_1 and E_2 are S-sets and $I = \overline{j(E)}$, then by Lemma 3,

$$\begin{aligned}\overline{j(E)} = I = \overline{I + J_1} \cap \overline{I + J_2} &= \overline{I + k(E_1)} \cap \overline{I + k(E_2)} \\ &= k(E_1) \cap k(E_2) = k(E),\end{aligned}$$

that is, E is an S-set. Conversely, if E is an S-set and $I = \overline{j(E)}$, then $\overline{j(E)} = I = \overline{I + J_1} \cap \overline{I + J_2} = J_1 \cap J_2$ and $k(E) = k(E_1) \cap k(E_2) = \overline{j(E)}$ by assumption. Hence, by Lemma 3, $J_1 = k(E_1)$, and $J_2 = k(E_2)$.

5. COROLLARY. *Let F be a spectral set and let E be a closed subset of F. If the relative boundary of E as a subset of F is a Ditkin set, then E is a spectral set.*

6. REMARK. The first half of Theorem 4–that the union of two S-sets which intersect in a Ditkin set is an S-set–was obtained for $A(G)$ by Herz [2, p. 228], and also, essentially, by Calderón [1, p. 3]. For the more general Banach algebra case, Reiter [6, p. 557] proved Lemma 3 and Theorem 4 in the case where the two sets were disjoint.

Our result, as well as that of [9], was suggested by a study of Reiter's paper [6]. Saeki [8, p. 551] gives an elegant proof in a Banach-algebraic setting of a result which is more general than the first half of Theorem 4. It is not difficult to see that our proofs of Lemma 3 and Theorem 4 can be modified slightly to prove the more general results, Lemma 3′ and Theorem 4′.

LEMMA 3′. *Let E_1 and E_2 be closed subsets of G, let $E = E_1 \cup E_2$, and suppose that there is a Ditkin set $C \subset E$ such that the intersection of the boundaries of E_1, E_2 and of E is contained in C. Let I be a closed ideal in $A(G)$ with $Z(I) = E$. Then I can be uniquely expressed in the form $I = I_1 \cap I_2$, where $I_i = \overline{I + J_i}$, and $J_i = \overline{j(E_i)}$ $(i = 1, 2)$. This form is unique in the sense that if $Z(I'_1) = E_1$, $Z(I'_2) = E_2$, and $I = I'_1 \cap I'_2$, then $I'_1 = I_1$ and $I'_2 = I_2$.*

THEOREM 4′. *Let E_1 and E_2 be closed subsets of G, let $E = E_1 \cup E_2$, and suppose there is a Ditkin set $C \subset E$ such that the intersection of the boundaries of E_1, E_2 and of E is contained in C. Then E is a spectral set if and only if both E_1 and E_2 are spectral sets.*

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