UNIFORM DISTRIBUTION ON LOCALLY COMPACT GROUPS

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ABSTRACT. In this paper we give a short proof of the existence of uniformly distributed sequences in separable locally compact groups, a result first stated by Benzinger [1]. The existence of u.d. sequences in separable compact groups was proved by Veech [7] whose results also imply the above result. We also prove the existence of well-distributed sequences.

H. Weyl's classical notion of uniform distribution admits an extension to arbitrary locally compact groups [6, Chapter 4, §5].

The main result of this paper is the following Theorem (compare also [7, Definition, p. 125] and [1, "key lemma", p. 150]).

THEOREM. *Let G be a locally compact group, C a class of nontrivial, continuous, unitary, irreducible, finite-dimensional representations of G . If there exists a countable subset F of G such that $U(F)$ is dense in $U(G)$ for each $U \in C$ (in the usual topology), then there exists a sequence (x_n) such that: For each $U \in C$ and any $\epsilon > 0$, there exists $M = M(U, \epsilon)$ such that*

$$\left\| \frac{1}{N} \sum_{n=k+1}^{n=k+N} U(x_n) \right\| < \epsilon \quad \text{for all } N \geq M, \quad k = 0, 1, 2, \dots$$

($\|\cdot\|$ = Hilbert-Schmidt norm).

PROOF. Let $B_1, B_2, \dots$ be an enumeration of all finite subsets of elements of F . If $B = (x_1, \dots, x_m)$ put

$$(B, U) = \frac{1}{m} \left\| \sum_{n=1}^m U(x_n) \right\|.$$

Using the fact that $U(x) = (u_{i,j}(x))$ where the $u_{i,j}$ may be considered as continuous functions on the Bohr compactification of G such that $\int u_{i,j}(x) dx = 0$ ("dx" denotes Haar measure), or using [3, Theorem 3.8.4], we obtain

$$(1) \quad \inf_n (B_n, U) = 0 \quad \text{for each } U \in C.$$

If $B = (x_1, \dots, x_m)$, $D = (y_1, \dots, y_k)$ put

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$$BD = (x_1 y_1, x_1 y_2, \dots, x_1 y_k, x_2 y_1, \dots, x_m y_k).$$

Considering the relations $(aB, U) = (Ba, U) = (B, U)$ (where $aB = (ax_1, \dots, ax_m)$, $a \in G$), $\|AV\| = \|VA\| = \|A\|$ for unitary V and arbitrary A , and $BD = (x_1 D, x_2 D, \dots, x_m D)$, and also using the fact that BD represents the same set as $(By_1, By_2, \dots, By_k)$, we obtain

$$(2) \quad (BD, U) \leq (B, U), \quad (BD, U) \leq (D, U) \quad \text{for all } U.$$

Replacing the block B_n by a block $z_n B_n$ we may assume that $B_n = (1, \dots)$. If we define A_m successively by $A_1 = B_1$, $A_{m+1} = B_{m+1} A_m$ then we have $A_1 \subseteq A_2 \subseteq \dots$, and obtain a sequence (x_n) such that the block consisting of the first K_m (= number of elements of A_m) terms coincides with A_m .

Given $\varepsilon > 0$, $U \in C$, then by (1) and (2) for a suitable m we have $(A_m, U) < \varepsilon/2$. We have $(x_1, x_2, \dots, x_n, \dots) = (A_m, z_2 A_m, z_3 A_m, \dots, z_j A_m, \dots)$ for suitable z_j 's. Therefore any sequence $(x_{k+1}, \dots, x_{k+N})$ can be divided into " A_m -like" blocks with the exception of at the most K_m terms. If U has dimension r ($\|U(x)\| = \sqrt{r}$ for all x in G) then we obtain for $N \geq 2K_m r^{1/2}/\varepsilon = M(U, \varepsilon)$,

$$\left\| \sum_{n=k+1}^{k+N} U(x_n) \right\| \leq K_m \sqrt{r} + (N/K_m) K_m (A_m, U) < N\varepsilon.$$

Q.E.D.

CONSEQUENCES. If G is a locally compact group then a sequence (x_n) is called uniformly distributed in G if for every closed normal subgroup such that G/H is compact, the sequence $(x_n H)$ is u.d. in G/H (Rubel [6, Chapter 4, Definition 5.2]). G is called K -separable if there exists a countable subset F such that FH is dense in G/H for any H as above. The following result of Benzinger [1, Theorem 3.6] is an immediate consequence of our Theorem and Weyl's criterion [6, Chapter 4, Theorem 5.5].

COROLLARY 1. *A locally compact group possesses a u.d. sequence if and only if G is K -separable.*

Corollary 1 is also a consequence of a deep result of Veech [7, Theorem 3]. He proves the existence of a sequence of positive integers (r_n) (uniformly distributed sequence generator) such that for any compact group K and any sequence (y_n) in K which is contained in no proper closed subgroup, the sequence $x_n = y_{r_1} y_{r_2} \dots y_{r_n}$ is u.d. in K . If G is K -separable there exists a sequence (y_n) such that $(y_n H)$ is dense in G/H , whenever G/H is compact; and the generated sequence (x_n) is u.d. in G .

If G is compact then (x_n) is called well distributed if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x_{n+k}) = \int f(x) dx$$

for any continuous f , uniformly in k (see [6, Chapter 4, Definition 1.1, Lemma 1.1]). If N denotes the intersection of the kernels of all continuous, unitary, finite-dimensional representations of G ("von Neumann-kernel"), and if $b: G \rightarrow bG$ denotes the Bohr homomorphism from G into the Bohr compacti-

fication, then we obtain by our Theorem and by Weyl's criterion for well-distributivity [6, Chapter 4, Corollary 1.3]

COROLLARY 2. *If G/N is separable then there exists a sequence (x_n) in G such that $b(x_n)$ is well distributed in $b(G)$.*

Both corollaries (for u.d. sequences) were proved in [2] for Abelian groups. A sequence x_n in G such that $b(x_n)$ is uniformly distributed in $b(G)$ is called Hartman u.d. We note that the construction of a Hartman u.d. sequence given in [2] actually yields a well-distributed sequence although the concept of well-distribution is not considered in that paper.

We mention the question of when the existence of a Hartman u.d. sequence implies the separability of G/N . For Abelian groups where, of course, $G/N = G$, this implication follows from Proposition 2.4 and Theorem A (Kakutani) of [2]. If G is almost connected, that is G/G_e is compact where G_e is the connected component of the identity, then we see that the existence of a Hartman u.d. sequence implies the separability of G/N by using the fact that G/N contains an open subgroup (of finite index) isomorphic to $R^n \times K$, K compact (see [4, Theorem 2.9(2)] and [5, VII, §2, 7.2.1]). We do not know, in general, whether there exists G such that G/N is nonseparable and yet G admits a Hartman u.d. sequence.

We note in closing that if G is Abelian and separable and if the set C in our Theorem is, in addition, compact (as a subset of $\hat{G}$), then the index $M(U, \epsilon)$ may be chosen independent of $U \in C$. A similar statement holds in the non-Abelian case.

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