A SHARP CONDITION FOR UNIVALENCE IN EUCLIDEAN SPACES

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ABSTRACT. Let $B \subset E^k$ be a ball. It is shown that if $f: B \rightarrow E^k$ is a local homeomorphism for which the infinitesimal change in length is bounded above by M and for which the infinitesimal change in volume is bounded below by m^k , where $M/m \leq 2^{1/k}$, then f is univalent. This result is numerically sharp.

Throughout, G will be an open set in E^k , the k -dimensional Euclidean space, $f: G \rightarrow E^k$ will be a local homeomorphism and $0 < m < M$ will be real numbers. We denote by $Q(G, m, M)$ the set of all f which satisfy the following two conditions for all $x \in G$:

$$(1) \quad M \geq \limsup_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|},$$

$$(2) \quad m \leq \liminf_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|}.$$

John [2], [3] calls such f quasi-isometric. As a slight improvement of a result of John [3], the author [1] showed that if $M/m \leq (4 + 2\pi)/(4 + \pi) = 1.439 \dots$, then all f in $Q(B, m, M)$ are univalent, where B is a ball. It seems reasonable to conjecture that this constant can be replaced by 2, but this appears difficult. Here, instead of approaching this conjecture by trying to improve the result of [3] still further, we shall consider a different class of mappings for which it is possible to prove a numerically sharp theorem. The result we get would follow if the conjecture were established. Our argument is based on the standard proof of the isoperimetric inequality $L^2 \geq 4\pi A$ on minimal surfaces of least area.

We define $Q'(G, m, M)$ to be the set of all f for which (1) together with

$$m^k \leq \liminf_{r \rightarrow 0} \frac{\mu(f(B(x, r)))}{\mu(B(x, r))}$$

hold for all $x \in G$, where μ denotes Lebesgue measure and $B(x, r)$ is the ball of radius r and center x . Our result is the following

THEOREM. *If $f \in Q'(B(x, r), m, M)$ and $M/m \leq 2^{1/k}$, then f is univalent. Furthermore, $2^{1/k}$ cannot be replaced by any larger constant.*

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Before proving this theorem we discuss the relationship between $Q'(G, m, M)$ and $Q(G, m, M)$. Obviously, $Q(G, m, M) \subset Q'(G, m, M)$. We show that

$$(3) \quad Q'(G, m, M) \subset Q(G, m^k/M^{k-1}, M).$$

The obvious Vitali covering argument shows that if $f \in Q'(G, m, M)$ is a homeomorphism on G , then $\mu(f(G)) \geq m^k \mu(G)$. If $X \subset E^k$, $B(X, \epsilon)$ will be the open ϵ -neighborhood of X . For $a, b \in E^k$, $[a, b]$ will denote the line segment connecting a and b . Furthermore, $s_k = \mu(B(x, 1))$. We need the following fact proved by John [2]:

LEMMA 1. *If $h: G \rightarrow E^k$ satisfies (1) for all $x \in G$ and if $[a, b] \subset G$, then $\|h(b) - h(a)\| \leq M\|b - a\|$.*

We now prove (3). We must show that for $f \in Q'(G, m, M)$ the $\liminf$ in (2) is bounded below by m^k/M^{k-1} for all $x \in G$. This being a local property, we may assume that f is one-to-one in G . Let $r > 0$ be such that $B(f(x), r) \subset f(G)$. Let $y \in J = f^{-1}(B(f(x), r/2))$ and $0 < \epsilon < r/2$. Let $T(\epsilon) = B([f(x), f(y)], \epsilon)$. If $z \notin f^{-1}(T(\epsilon))$, then Lemma 1 implies that z must be at least ϵ/M from $f^{-1}([f(x), f(y)])$ which is a curve joining x and y . Thus $f^{-1}(T(\epsilon))$ contains an (ϵ/M) -neighborhood of this curve. Hence $\mu(f^{-1}(T(\epsilon)))$ is bounded below by $\|x - y\|_{s_{k-1}}(\epsilon/M)^{k-1}$. But $\mu(T(\epsilon))$ is given by

$$\|f(x) - f(y)\|_{s_{k-1}} \epsilon^{k-1} + s_k \epsilon^k.$$

Since $\mu(T(\epsilon)) \geq m^k \mu(f^{-1}(T(\epsilon)))$, we have

$$\|f(x) - f(y)\|_{s_{k-1}} \epsilon^{k-1} + s_k \epsilon^k \geq m^k \|x - y\|_{s_{k-1}} (\epsilon/M)^{k-1}.$$

Allowing $\epsilon \rightarrow 0$, we see that $\|f(x) - f(y)\|$ is bounded below by $(m^k/M^{k-1}) \cdot \|x - y\|$, which gives (3) since $y \in J$ is arbitrary.

We note in passing that the m^k/M^{k-1} appearing in (3) cannot be replaced by any larger number as is shown by the linear transformation whose matrix with respect to some orthonormal basis of E^k is diagonal with entries $m^k/M^{k-1}, M, \dots, M$.

Before beginning the proof of our theorem we need some terminology and facts. In what follows we shall not distinguish explicitly between a curve or surface as such and the set of points that lie on it. This should cause no confusion. Arc length and plane measure will be denoted by λ and μ_2 , respectively. For brevity we write $B(r) = B(0, r)$ and denote closed balls with a bar. Also, $D = \bar{B}(1)$ in E^2 . For a closed curve C in E^k we say that the surface given by $\psi: D \rightarrow E^k$ spans C if $\psi|_{\partial D}$ is a parametrization of C and we say that C links with $A \subset E^k$ if $C \cap A = \emptyset$ and every surface which spans C intersects A .

LEMMA 2. *If $h: G \rightarrow E^k$ satisfies (1) in G and if C is a rectifiable curve lying in G , then $\lambda(h(C)) \leq M\lambda(C)$.*

This is an obvious consequence of Lemma 1.

LEMMA 3. *Let $k \geq 3$. Let B be an open ball in E^k and let $h: \bar{B} \rightarrow E^k$ be a*

continuous mapping for which $h|_B$ is a homeomorphism. Let $h(a) = h(b)$, where $a \neq b$ and $a, b \in \partial B$. Let C' and C'' be two simple curves which connect a and b in B and on ∂B , respectively. Let $A \subset B$ be compact. If $C' \cup C''$ links with A , then $h(C')$ links with $h(A)$.

This lemma is intuitively clear and a simple proof may be found in [1].

Let $H \subset E^k$, $k \geq 3$, be any closed half plane. Let $x = (x_1, \dots, x_k)$ be a Cartesian coordinate system for which $H = \{x: x_2 \geq 0 \text{ and } x_j = 0 \text{ for } j \geq 3\}$. We define $W_H: E^k \rightarrow H$ by $W_H(x) = (x_1, (x_2^2 + \dots + x_k^2)^{1/2}, 0, \dots, 0)$.

LEMMA 4. Let $f \in Q'(G, m, M)$ be one-to-one in G and let the closure of X be contained in G . Assume that $f(X)$ is an open subset of some plane P lying in E^k . Then $\mu_2(f(X))$ is bounded below by $(m^k/M^{k-2})\mu_2(W_H(X))$.

PROOF. First of all, we can assume that $W_H(X)$ lies at a positive distance from L , the edge of H , since obviously $\mu_2(W_H(X - \bar{B}(L, \epsilon))) \rightarrow \mu_2(W_H(X))$ as $\epsilon \rightarrow 0$. Secondly, since the inverse of a homeomorphism h in $Q(G, m, M)$ is in $Q(h(G), 1/M, 1/m)$, we see by (3) that

$$f^{-1} \in Q(f(G), 1/M, M^{k-1}/m^k).$$

Also, W_H is Lipschitz continuous with constant 1. These two facts imply that if $A \subset P$ and $\mu_2(A) = 0$, then $\mu_2(W_H(f^{-1}(A))) = 0$. Since $f(X) = A \cup R$, where $\mu_2(A) = 0$ and R is a countable disjoint union of squares, we may assume that $f(X)$ is itself a square. Our first assumption yields

$$\mu(B(X, \epsilon)) \geq s_{k-2} \epsilon^{k-2} \mu_2(W_H(X)) - o(\epsilon^{k-2}).$$

However, for ϵ small $f(B(X, \epsilon))$ is contained in $B(f(X), M\epsilon)$ whose measure is $s_{k-2}(M\epsilon)^{k-2} \mu_2(f(X))$ to within $o(\epsilon^{k-2})$ by our other assumption. By the fact mentioned just after (3),

$$s_{k-2}(M\epsilon)^{k-2} \mu_2(f(X)) \geq m^k s_{k-2} \epsilon^{k-2} \mu_2(W_H(X)) - o(\epsilon^{k-2}).$$

Upon dividing by $s_{k-2} \epsilon^{k-2}$ and letting $\epsilon \rightarrow 0$ we obtain our result.

We now begin the proof of the theorem. Clearly we may assume that $x = 0$. We suppose that $f \in Q'(B(r), m, M)$ is not univalent and show that this implies that $M/m > 2^{1/k}$. First we do the case $k \geq 3$. The case $k = 2$ may be handled with minor modifications or may be deduced from the higher dimensional case as we show later. Let $r_0 = \inf\{s: f \text{ is not univalent in } B(s)\}$. Then $r > r_0 > 0$ and there are two distinct points a and b on $\partial B(r_0)$ for which $f(a) = f(b)$. We have $r_0 > 0$, because the mapping is a local homeomorphism. The existence of the two points can be justified as follows: There exist two sequences $\{a_n\}$ and $\{b_n\}$, which can be assumed convergent, such that $a_n \neq b_n$, $f(a_n) = f(b_n)$ and $a_n, b_n \in B(r_0 + 1/n)$. If a and b are the limits of these sequences, then $a \neq b$, since f is a local homeomorphism. Also, we have $a, b \in \bar{B}(r_0)$ and the definition of r_0 implies that $a, b \in \partial B(r_0)$.

Let P be the plane containing a, b and 0 . Let L be the line in P through 0 parallel to $[a, b]$ and let $H \subset P$ be the closed half plane with edge L containing a and b . Let c be the midpoint of $[a, b]$ and let $K = \bar{B}(c, \|a - b\|/2) \cap (H - L) \cap B(r_0)$. Let $C = \partial K$ and let C' denote the part of C which connects a and b inside $B(r_0)$. Simple geometric considerations show that

$$(4) \quad (\lambda(C'))^2 < 2\pi\mu_2(K).$$

Since $f(a) = f(b)$, $f(C')$ is a closed curve. Evidently, for any $\varepsilon > 0$ there are $n = n(\varepsilon)$ consecutive points $f(a) = p_0, p_1, \dots, p_n = f(a)$ on $f(C')$ such that the piece of $f(C')$ between p_{i-1} and p_i is contained in $B(p_i, \varepsilon)$, for $1 \leq i \leq n$. Let $C(\varepsilon)$ denote the closed piecewise linear curve formed by connecting these n points by line segments in order.

For $x \in H$, let $S(x)$ be the $(k-2)$ -dimensional sphere $W_H^{-1}(x)$. If $x \in K$, then C links with $S(x)$. Were this not the case, there would be a surface spanning C but disjoint from $S(x)$. Applying W_H to this surface we get a surface spanning C and lying in H but not containing x . This, however, is impossible since $x \in K$, the interior of C . Thus Lemma 3 implies that for $x \in K$, $f(C')$ links with $f(S(x))$. There is a $\delta = \delta(\varepsilon) > 0$, which goes to 0 when ε goes to 0, such that $C(\varepsilon)$ links with $f(S(x))$ for each $x \in K$ whose distance from C is at least δ . This follows since if one continuously deforms a closed curve which links with a set A without touching A , then the resulting curve links with A .

For $p \in E^k$ we denote by $d(\varepsilon, p)$ the surface spanning $C(\varepsilon)$ generated by the line segments $[p, x]$, $x \in C(\varepsilon)$. This surface is a union of triangles. We denote by $\tau(p)$ the sum of the angles $p_{i-1}pp_i$, $1 \leq i \leq n$. Here angles lie between 0 and π . The function τ is well defined and continuous for $p \neq p_0, p_1, \dots, p_n$ and approaches 0 as $\|p\| \rightarrow \infty$. Let $\psi(p)$ be the same sum except that $2 \leq i \leq n-1$. Clearly, ψ is even continuous at p_0 . If $\psi(f(a)) \leq 2\pi$, then we can "lay the surface $d(\varepsilon, f(a))$ out" in E^2 by placing the triangle $p_0p_1p_2$ in E^2 , then placing $p_0p_2p_3$ next to it so that the sides $[p_0, p_2]$ of both triangles coincide, etc. In this case we set $q = f(a)$. If, on the contrary, $\psi(f(a)) > 2\pi$, then the facts that $\tau(p) \geq \psi(p)$, $\psi(p) \rightarrow \psi(f(a)) > 2\pi$ as $p \rightarrow f(a)$ and $\tau(p) \rightarrow 0$ as $\|p\| \rightarrow \infty$ imply, by the intermediate value theorem, that there is a $q \notin C(\varepsilon)$ for which $\tau(q) = 2\pi$. In this case, the surface $d(\varepsilon, q)$ can be "laid out" in E^2 by placing the triangles $p_{i-1}qp_i$ so that the sides $[p_i, q]$ of $p_{i-1}qp_i$ and p_iqp_{i+1} coincide, for $1 \leq i \leq n$, where $p_{n+1} = p_1$.

In either case the "image" of $d(\varepsilon, q)$ is a polygon T and ∂T is the "image" of a part of $C(\varepsilon)$. (If $\psi(f(a)) \neq 2\pi$ and there are no degenerate triangles, then ∂T will be the "image" of all of $C(\varepsilon)$.) Moreover, the area of T is the sum of the areas of the triangles making up $d(\varepsilon, q)$. If we now apply Lemma 4 to each of these triangles, we see that $\mu_2(T)$ is bounded below by $(m^k/M^{k-2}) \cdot (\mu_2(K) - (\delta))$, since the image under W_H of the set $f^{-1}(f(B(r_0)) \cap d(\varepsilon, q))$ contains all the points of K whose distance from C is at least δ . The usual isoperimetric inequality gives us that $(\lambda(\partial T))^2 \geq 4\pi\mu_2(T)$. But

$$\lambda(\partial T) \leq \sum_{i=1}^n \|p_i - p_{i-1}\| \leq \lambda(f(C')) \leq M\lambda(C'),$$

the last inequality following from Lemma 2. Thus $M^2(\lambda(C'))^2 \geq 4\pi(m^k/M^{k-2})\mu_2(K) - O(\delta)$. Allowing ε to go to 0 and substituting (4), we obtain the desired inequality.

We now discuss the two dimensional case. For $k \geq 2$ we view E^k as $E^2 \times E^{k-2}$. Let $U \subset E^2$ and $V \subset E^{k-2}$ and let $g: U \rightarrow E^2$ and $h: V \rightarrow E^{k-2}$. Define the mapping $g \times h$ by $g \times h(x, y) = (g(x), h(y))$ for $(x, y) \in U \times V$. Let B be an open disc in E^2 and $f \in Q'(B, m, M)$. Let $h(y) = My$

for $y \in E^{k-2}$. It is clear that $f \times h \in Q'(B \times E^{k-2}, m', M)$, where $m' = (m^2 M^{k-2})^{1/k}$. For $k \geq 3$, by what we have already proved $f \times h$ will be univalent in any ball contained in $B \times E^{k-2}$ if $M/m' \leq 2^{1/k}$, that is, if $M/m \leq \sqrt{2}$. Since f is univalent in B if $f \times h$ is univalent in some set containing $B \times \{0\}$, we obtain the result for $k = 2$ once we have it in the higher dimensional case.

Finally, we show that the result is numerically sharp. Let E^2 be given polar coordinates (r, θ) . Let U be the right half plane, $-\pi/2 < \theta < \pi/2$. Define $g(r, \theta) = (r, (2 + \beta)\theta)$. If h is as in the preceding paragraph with $M = 2 + \beta$, we see that $g \times h \in Q'(U \times E^{k-2}, m, 2 + \beta)$, where $m^k = (2 + \beta)^{k-1}$. For $\beta > 0$, $g \times h$ is not one to one in the k -dimensional ball $B(x, 1)$, where $x = (1, 0, \dots, 0)$. Since $M/m \rightarrow 2^{1/k}$ as $\beta \rightarrow 0$, the proof is complete.

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