THE LOGARITHMIC CENTER OF A PLANAR REGION

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ABSTRACT. Given a bounded region S in the complex plane, let $f(\beta) = \int_S \log |z - \beta| d \text{ area}$ for β any complex number. A *logarithmic center* of S is an α which minimizes $f(\beta)$. When is α unique?

CONJECTURE. If S is convex then α is unique.

THEOREM. If S is convex and symmetric about some line, then α is unique.

0. Introduction. Consider a “reasonable” closed bounded region S in the complex plane.

DEFINITION. A *logarithmic center* of S is an α which minimizes the logarithmic potential

$$(1) \quad f_S(\beta) = \int_S \log |z - \beta| d \text{ area}.$$

The following application to number theory motivates the problem. If α_1 is a logarithmic center of S and α_2 is not, then for sufficiently large x , $x(S - \alpha_1)$ will contain more prime Gaussian integers than $x(S - \alpha_2)$. This is seen by comparing the estimate (2) for Gaussian primes in $x(S - \alpha_i)$ ($i = 1, 2$).

$$(2) \quad \text{Li}(S) = \frac{2}{\pi} \int_{S \cap \{z: |z| > 2\}} \frac{d \text{ area}}{\log |z|} \quad [1], [2].$$

For the general region S , there need not be a unique logarithmic center. We give two examples.

The shaded region has a logarithmic center in each end.

Every point inside the inner of two concentric circles is a logarithmic center of the shaded region.

For certain regions such as a disc, a rectangle, or an equilateral triangle, the symmetry suggests that the geometric center should also be the (unique) logarithmic center.

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In §1 we give sufficient conditions for a region to have a unique logarithmic center, and state a more general conjecture.

1. **Conjecture.** If S is convex, S has a unique logarithmic center.¹

THEOREM 1. *If S is convex and has a line of mirror symmetry, S has a unique logarithmic center.*

THEOREM 2. *If S is convex, S_1 and S_2 concentric disks with radii in ratio 10:11 and $S_1 \subseteq S \subseteq S_2$ then S has a unique logarithmic center.*

PROOF OF THEOREM 1. We first make several preliminary observations, none of which is at all deep. For the most part, the proofs of these are omitted or drawn sketchily. We shall sometimes write f in place of f_S . Here $S - \beta$ denotes the translate $S - \beta = \{x - \beta : x \in S\}$.

1.

$$\frac{\partial f}{\partial x}(\beta) = - \int_{S-\beta} \frac{x}{x^2 + y^2} dx dy, \quad \text{and} \quad \frac{\partial f}{\partial y}(\beta) = - \int_{S-\beta} \frac{y}{x^2 + y^2} dx dy.$$

2. Let ∂S be the positively oriented boundary of S . Then

$$\frac{\partial f}{\partial x}(\beta) + i \frac{\partial f}{\partial y}(\beta) = i \int_{\partial S} \log |z - \beta| dz.$$

REMARK. $\partial f / \partial x + i \partial f / \partial y$ corresponds to the gradient of f .

3. Let $g(\beta) = (\partial f / \partial x + i \partial f / \partial y)(\beta)$. Then

$$\frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y} = -i \int_{\partial S} \frac{dz}{z - \beta} = -i[0 \text{ if } \beta \notin S, 2\pi i \text{ if } \beta \in S^0].$$

4. If $\beta \notin S$, f is harmonic at β . If $\beta \in S^0$, $(\partial^2 f / \partial x^2 + \partial^2 f / \partial y^2)(\beta) = 2\pi$. This follows immediately from 3.

5. If α is a logarithmic center of S , and if $c, d \in \mathbb{C}$, then $c\beta + d$ is a logarithmic center of $cS + d$. The same holds for stationary points of f_S .

6. There exists at least one logarithmic center of S .

PROOF. $f(\beta)$ is a continuous function which grows without bound as $|\beta| \rightarrow \infty$. Therefore a minimum exists.

7. Suppose a straight line M passes through a closed convex S , dividing S into two parts S_1 and S_2 . If the reflection of S_1 across M is a proper subset of S_2 , then no point of S_1 is a stationary point of f_S .

PROOF. For $\alpha \in S_1$, establish coordinate axes through α so that the y axis is parallel to M . The y axis divides S into two parts S'_1 and S'_2 so that $S'_1 \subseteq S_1$ and $S_2 \subseteq S'_2$.

The reflection of S'_1 across the y axis is a proper subset of S'_2 , and so

$$\frac{\partial f_S}{\partial x}(\alpha) = - \int_S \frac{x}{x^2 + y^2} = - \int_T \frac{x}{x^2 + y^2} \neq 0,$$

where $T = S'_2 \setminus$ the reflection of S'_1 , because the integrand is positive and the area of integration nonzero.

¹ This is proved in an unpublished manuscript which the author will send to persons requesting it.

DEFINITION. A saddle point of f_S (or "of f " or "of S ") is a stationary point of f which is not a strict local minimum (or maximum, but this is impossible as f is subharmonic).

8. For convex S , let m_0 denote the number of minima of f_S , and m_1 the number of saddle points. Then $m_0 = m_1 + 1$.

REMARK. This is the simplest case of a general theory due to Morse linking the number of maxima, minima, and saddle points to the genus of a surface. See [3].

9. If $\alpha \in S$ is a saddle point of S , and if the disc of radius 1 about α is in S , then there exists $\beta \in S$ such that $|\alpha - \beta| \geq e^{\pi/2}$.

PROOF. Establish xy coordinates through α so that $(\partial^2 f / \partial x^2)(\alpha) \leq 0$, $(\partial^2 f / \partial x \partial y)(\alpha) = 0$. $f_S(\alpha) = f_{(S \setminus \text{disc})}(\alpha) + f_{\text{disc}}(\alpha)$. By fact 4,

$$(\partial^2 / \partial x^2 + \partial^2 / \partial y^2) f_{S \setminus \text{disc}}(\alpha) = 0,$$

and by fact 4 and symmetry (or direct calculation)

$$(\partial^2 / \partial x^2) f_{\text{disc}}(\alpha) = (\partial^2 / \partial y^2) f_{\text{disc}}(\alpha) = \pi.$$

Now

$$\frac{\partial^2}{\partial x^2} f_{S \setminus \text{disc}}(\alpha) = - \int_{S \setminus \text{disc}} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy$$

(this is legitimate if one stays away from the singularity at α) so

$$\frac{\partial^2}{\partial x^2} f_S(\alpha) = \pi - \int_{S \setminus \text{disc}} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy.$$

Since $(\partial^2 / \partial x^2) f_S(\alpha) \leq 0$, we have

$$\int_{S \setminus \text{disc}} \frac{x^2 + y^2}{(x^2 + y^2)^2} dx dy \geq \pi.$$

Converting to polar coordinates, and neglecting the region where $x^2 - y^2 < 0$, we have

$$2 \int_{-\pi/4}^{\pi/4} \int_1^{\partial S} \frac{1}{r} \cos 2\theta dr d\theta \geq 4 \Rightarrow \int_1^T \int_{-\pi/4}^{\pi/4} \frac{\cos 2\theta}{r} d\theta \geq \frac{\pi}{2}$$

where $T = \sup_{\beta \in S} |\alpha - \beta|$. Thus $\log T \geq \pi/2$ and $T \geq e^{\pi/2}$ as claimed.

Now suppose S is convex, symmetric about L . By 7 and 8, if S has two logarithmic centers then there exists a saddle point p on L , between the two centers, which also lie on L . Establish xy coordinates through p with L the x axis. For g above L , by the reflection argument of fact 7 with M through q parallel to L , we have $(\partial f / \partial y)(q) > 0$. Similarly for q below L , $(\partial f / \partial y)(q) < 0$. Thus $(\partial^2 / \partial y^2) f(p) \geq 0$. By symmetry, or since $\partial f / \partial y \equiv 0$ on L , we have $\partial^2 f / \partial x \partial y \equiv 0$ on L . Therefore $(\partial^2 / \partial x^2) f(p) \leq 0$, since otherwise p would be a minimum, not a saddle point.

REMARK. Actually for the symmetric case we only need p on the x axis stationary, with $\partial^2 f / \partial x^2 \leq 0$.

Now for $(x, y) \in \partial S$ let $\vec{n}(x, y)$ be the unit normal vector outward from the boundary of S . $\vec{n}$ is defined except on a set of measure 0. Let $\mu(x, y)$ be the x component of $\vec{n}(x, y)$, and $\nu(x, y)$ the y component. Then with a translation of coordinate axes so that $p = 0$, we have:

$$(*) \quad \frac{\partial^2 f}{\partial x^2}(p) = \int_{\partial S} \frac{x\mu(x, y)}{x^2 + y^2} d \text{ arc length}.$$

Assume without loss of generality that the leftmost point of S is at least as far as the rightmost from p . Since $(\partial f / \partial x)(p) = 0$, the reflection of $(S \cap \text{left half plane})$ across the y axis does not properly contain $(S \cap \text{left half plane})$. Thus for points $q = (x, y)$ of $(S \cap \text{left half plane})$ whose reflection lies farther to the right than any point of S , we have $\mu(q) < 0$, and so $x\mu(q)/(x^2 + y^2) > 0$.

Let a, b be respectively the left and right endpoints of $S \cap (x \text{ axis})$. Let U be the portion of ∂S whose x coordinates are between a and $-b$, and let $V = \partial S \setminus U$. Let $y(x)$ be the nonnegative number such that $(x, y(x)) \in \partial S$, for $-b \leq x \leq b$.

Then the

$$\begin{aligned} \text{integral of } (*) &= \int_U \frac{x\mu(x, y)}{x^2 + y^2} d \text{ arc length} + \int_V \frac{x\mu(x, y)}{x^2 + y^2} d \text{ arc length} \\ &\geq \int_U \frac{x\mu(x, y)}{x^2 + y^2} d \text{ arc length} + 2 \int_{-b}^b \left(\frac{-xy'(x)}{x^2 + y^2(x)} \right) dx. \end{aligned}$$

(Inequality occurs if and only if V has a vertical segment.) As we have noted, the integrand is positive on U . The second term

$$= -2 \int_0^b \left(\frac{xy'(x)}{x^2 + y^2(x)} - \frac{xy'(-x)}{x^2 + y^2(-x)} \right) dx.$$

We shall see that the integrand here is ≤ 0 , and $= 0$ only when $y'(x) = y'(-x) = 0$. Now either (1) U is of positive measure or (2) $y'(x)$ is not $\equiv 0$ on $(-b, b)$ or (3) S is a rectangle, and p is the center, hence not a saddle point. In cases (1) or (2), we have $(\partial^2 f / \partial x^2)(p) > 0$, so again p is not a saddle point, a contradiction. We now check that the second integral ≤ 0 and thus prove Theorem 1.

If $y'(x) \geq 0$, then because S is convex, $y(x) \geq y(-x)$ and $y'(-x) \geq y'(x)$, and so the integrand is ≤ 0 (< 0 unless $y'(x) = y'(-x) = 0$).

If $y'(-x) \leq 0$, then the same argument shows the integrand again ≤ 0 .

And if $y'(-x) \geq 0$ and $y'(x) \leq 0$ then both terms of the integrand are ≤ 0 . This completes the proof of Theorem 1.

PROOF OF THEOREM 2. We have concentric discs S_1 and S_2 , of radius 10 and 11 respectively, and $S_1 \subseteq S \subseteq S_2$. S is assumed convex.

REMARK. One can show there exist radii $m_1 \neq m_2$ such that if $S_1 \subseteq S \subseteq S_2$, S_i of radius m_i then S has a unique logarithmic center, whether or not S is convex. We do not go into this; the ideas are much the same, with computation taking the place of comparisons by reflection arguments.

Now assume for purposes of contradiction that f_S has two minima α_1 and

α_2 . Then from 8 there is a saddle point p , and from 7, $p \in S$.

Case I. Length of $\bar{O}p \geq 5$, where O is the common center of the discs. Construct the line M through p perpendicular to $\bar{O}p$. Let m_1 and m_2 be the two points of $\partial S \cap M$. Construct lines M_1 and M_2 through m_1, m_2 respectively and tangent to S_1 as in the figure, at t_1, t_2 .

The portion of S to the left of M lies between M_1 and M_2 , while S contains the quadrilateral $m_1 m_2 t_1 t_2$. Since $10^2 + 5^2 \geq 11^2$, M_1 and M_2 intersect to the left as indicated in the figure, and $(\text{left side of } M) \cap S$ reflects properly into $(\text{right side of } M) \cap S$. But by 7, p cannot even be stationary, a contradiction.

Case II. Length $\bar{O}p < 5$. Then $\sup_{\beta \in S} [\text{distance}(p, \beta)] < 16$, while the disc of radius 5 about p is in S . Thus by 9, p is not a saddle point since $16 < 5e^{\pi/2}$, a contradiction and Theorem 2 is proved.

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