SULLIVAN'S DE RHAM COMPLEX IS DEFINABLE IN TERMS OF ITS 0-FORMS¹

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ABSTRACT. It is proved that for a simplicial complex Sullivan's de Rham complex is definable in terms of its 0-forms.

1. Statement of the result. Sullivan [1] constructed, for every *simplicial complex* K , a *polynomial de Rham complex* AK and proved that this differential graded commutative algebra over the rationals Q contains much information about the rational homotopy type of K . We demonstrated recently [2] that this polynomial de Rham complex AK , and in fact already its *algebra of polynomial 0-forms* A^0K , contains *all* the information about the *whole* homotopy type of K . This result suggests the possibility of *recovering the whole de Rham complex* AK from its *algebra of 0-forms* A^0K , and the purpose of this note is to show that this indeed can be done. In more detail:

Let (DG) be the functor which (see §2) assigns to every *commutative algebra* Y (over Q) the "free-est" *differential graded commutative algebra* with Y in degree 0. (In algebraic geometry this is often denoted by $\Omega^0(y)$.) Then we have

THEOREM. *For every finite simplicial complex* K *the obvious (see §2) map* $p: (DG)A^0K \rightarrow AK$ *has the properties*

- (i) p is onto, and
- (ii) $x \in \ker p$ if and only if, for every finitely generated polynomial algebra P and every map $q: (DG)A^0K \rightarrow (DG)P$, one has $x \in \ker q$.

The rest of this note consists of a proof of this theorem. It uses some of the results of [2] and Bousfield-Gugenheim [1]. The latter reference contains a clear account of Sullivan's theory for simplicial complexes and its extension to simplicial sets.

2. Proof of the theorem. We first recall that, for a commutative algebra Y (over Q), the "free-est" differential graded commutative algebra $(DG)Y$ with Y in degree 0 is the *tensor product* of the *polynomial algebra* on symbols $[y]$ for all $y \in Y$ and the *exterior algebra* on symbols $[dy]$ for all $y \in Y$, *modulo the ideal* generated by the relations

$$\begin{aligned} [y + y'] &= [y] + [y'], & [d(y + y')] &= [dy] + [dy'], \\ [y \cdot y'] &= [y] \cdot [y'], & [d(y \cdot y')] &= [y] \cdot [dy'] + [y'] \cdot [dy], \end{aligned}$$

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$$[ry] = r[y], \quad [d(ry)] = r[dy]$$

for all $y, y' \in Y$ and $r \in Q$. For every differential graded commutative algebra X , there is precisely one map (of DG algebras) $p: (DG)x^0 \rightarrow X$ which induces the identity on X^0 .

PROOF OF PART (i) OF THE THEOREM. Choose an order for the vertices of K . Then K has an associated simplicial set (of which the nondegenerate simplices are in 1-1 correspondence with the simplices of K), which we will also denote by K (the ΔK of [1, §13]). Let Δ^n be a simplicial standard simplex that contains K as a subcomplex (here we use the fact that K "comes from a simplicial complex"). Then [1, §§1 and 2] one has a commutative diagram

$$\begin{array}{ccc} (DG)A^0\Delta^n & \xrightarrow[p]{\approx} & A\Delta^n \\ \downarrow & & \downarrow \\ (DG)A^0K & \xrightarrow[p]{} & AK \end{array}$$

in which the top map is an isomorphism and the map on the right is onto, and this clearly implies that *the bottom map is also onto*.

PROOF OF PART (ii) OF THE THEOREM. Consider the pair of adjoint contravariant functors

$$A: (\text{simplicial sets}) \leftrightarrow (DG \text{ algebras}) : F$$

of [1, §8]. The adjunction map $\text{Id} \rightarrow AF$ gives rise to a commutative diagram

$$\begin{array}{ccc} (DG)A^0K & \xrightarrow{h} & AF(DG)A^0K \\ \downarrow p & & \downarrow AFp \\ AK & \xrightarrow{k} & AFAK \end{array}$$

in which the map k is 1-1. Moreover it follows readily from the definitions of the functors A and F that $x \in \ker h$ if and only if, for every finitely generated polynomial algebra P and every map $q: (DG)A^0K \rightarrow (DG)P$, one has $x \in \ker q$. Thus it remains to show that the map AFp is an isomorphism.

Clearly the simplices of $F(DG)A^0K$, i.e. the maps of DG algebras $(DG)A^0K \rightarrow (DG)P$, are in 1-1 correspondence with the maps of algebras $A^0K \rightarrow P$. Furthermore every simplex of FAK , i.e. map of DG algebras $AK \rightarrow (DG)P$, induces by restriction a map of algebras $A^0K \rightarrow P$. It is not hard to see (using the fact that the map p is onto) that this correspondence is 1-1. To show that it is also onto, take a map of algebras $f: A^0K \rightarrow P$ and observe that [2, 3.7] f admits a factorization

$$A^0K \xrightarrow{A^0(\Delta x)} A^0\Delta r \xrightarrow{g} P$$

and thus f is the restriction of the composition

$$AK \xrightarrow{A(\Delta X)} A\Delta' \approx (DG)A^0\Delta' \xrightarrow{(DG)g} (DG)P.$$

Thus Fp , and hence AFp , is an isomorphism.

REMARK. Implicit in the above proof is the fact that, for a *simplicial complex* K , one has an *isomorphism* $F^0 A^0 K \approx FAK$ where F^0 is as in [2, 3.3]. Hence [2, 1.1] the *adjunction map* $K \rightarrow FAK$ is a *weak homotopy equivalence* for every (finite) *simplicial complex* K .

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