A NOTE ON THE METRIZATION OF γ -SPACES

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ABSTRACT. The class of quasi Nagata spaces, which strictly includes the class of wN -spaces, is defined. A closely related class of spaces, pseudo Nagata spaces, are also defined. It is shown that a Hausdorff γ -space is metrizable if and only if it is either a quasi Nagata space or a pseudo Nagata space.

In [5], R. Hodel introduced the class of wN -spaces, a generalization of the class of Nagata spaces, and showed that a Hausdorff γ -space is metrizable if and only if it is a wN -space. The purpose of this note is to introduce the class of quasi Nagata spaces, which strictly includes the class of wN -spaces, and to generalize Hodel's theorem by showing that a Hausdorff γ -space is metrizable if and only if it is a quasi Nagata space. We also introduce the class of pseudo Nagata spaces and show that a pseudo Nagata Hausdorff γ -space is metrizable. In the class of weakly first countable spaces [1, p. 129], [16, p. 15], the class of pseudo Nagata spaces properly contains the class of wN -spaces. Throughout the note, we let N denote the set of all positive integers.

A collection $G = \{g(x, n): x \in X, n \in N\}$ will be called a *graded system of open covers* for the space X if and only if the following two conditions are satisfied: (i) x is an element of $g(x, n)$ and $g(x, n)$ is an open subset of X for all x in X and n in N ; (ii) $g(x, n+1) \subset g(x, n)$ for all n and all x . If, in addition, $\{g(x, n): n \in N\}$ is a local base at x for each x in X , then G is called a *graded open base* for the space X . If $G = \{g(x, n)\}$ is a graded system of open covers for a space X and A is a subset of X , we let $g(A, n) = \bigcup \{g(a, n): a \in A\}$.

Nagata spaces are first countable stratifiable spaces [2]. In [4, Theorem 5] and independently in [9], it is shown that a space X is a Nagata space if and only if X has a graded open base $\{g(x, n)\}$ such that if A is a compact subset of X and B is closed and disjoint from A , then there exists some natural number n for which $g(B, n) \cap A = \emptyset$. Reversing the roles of A and B in this characterization of Nagata spaces leads to the notion of a γ -space. A graded open base $\{g(x, n): x \in X, n \in N\}$ is said to be a γ -basis for the space X if and only if whenever A is a compact set and V is an open neighborhood of A , then there exists a natural number n such that $g(A, n) \subset V$. A space X is said to be a γ -space provided that X has a γ -basis. The notion of a γ -space has recently arisen independently in several different places. Specifically, in [3] it is shown that the following classes of spaces are equivalent: co-Nagata spaces

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[11], γ -spaces [5], spaces having a coconvergent open neighborhood assignment [14], locally quasi-uniform spaces with countable bases [3], spaces admitting an α -metric that satisfies property π [13], and Nagata first countable spaces [6].

A topological space X is said to be a *wN-space* provided that X has a graded system of open covers $\{g(x, n): x \in X, n \in N\}$ which satisfies the following condition: if $\{x_n\}$ is a sequence in X and there exists a point x in X for which $g(x, n) \cap g(x_n, n)$ is nonempty for all $n \in N$, then the sequence $\{x_n\}$ has a cluster point.

A topological space X is said to be a *quasi Nagata space* if and only if X has a graded system of open covers $\{g(x, n): x \in X, n \in N\}$ such that if $\{x_n\}$ is a convergent sequence in X and $x_n \in g(y_n, n)$ for all $n \in N$, then the sequence $\{y_n\}$ has a cluster point.

A collection $G = \{g(x, n): x \in X, n \in N\}$ of subsets of X is called a *graded weak basis* for the space X provided that the following three conditions hold: (i) for all x in X and n in N , we have $x \in g(x, n)$; (ii) $g(x, n+1) \subset g(x, n)$ for all x and all n ; (iii) a subset B of X is closed if and only if whenever $x \in X - B$, then there exists an $n = n(x)$ in N such that $g(x, n) \cap B = \emptyset$. We define a space X to be a *pseudo Nagata space* provided that X has a graded weak base $\{g(x, n)\}$ such that the following implication holds: if $\{x_n\}$ is a convergent sequence in X and $x_n \in g(y_n, n)$ for all $n \in N$, then the sequence $\{y_n\}$ has a cluster point.

The motivation behind the concept of a pseudo Nagata space arises in part from Hodel's theorem and in part from Arhangel'skii's concept of a weak base [1, p. 129] and the fact that many well-known metrization theorems may be extended to a weak base setting, e.g., see [7], [8], [12], [15].

Let X be a wN-space. Then X has a graded system of open covers $\{g(x, n): x \in X, n \in N\}$ such that if $g(p, n) \cap g(x_n, n) \neq \emptyset$ for all $n \in N$, then $\{x_n\}$ has a cluster point. Let $y_i \in g(x_i, i)$ for all $i \in N$ and suppose that $\{y_i\}$ converges to a point y . We shall show that $\{x_i\}$ has a cluster point. Choose $n_i \geq i$ for all $i \in N$ such that $y_{n_i} \in g(y, i)$, where $n_i \leq n_{i+1}$. We have $y_{n_i} \in g(y, i) \cap g(x_{n_i}, n_i)$ for all i ; since $n_i \geq i$, we also have $g(y, i) \cap g(x_{n_i}, i) \neq \emptyset$ for all i . It follows that $\{x_{n_i}\}$ has a cluster point, that is, $\{x_i\}$ has a cluster point, and we have established the following:

REMARK 1. Every wN-space is a quasi Nagata space.

The following simple lemma will prove useful below.

LEMMA 1. Let $\{g(x, n): x \in X, n \in N\}$ be a graded system of open covers for a space X and let $\{h(x, n): x \in X, n \in N\}$ be a graded weak base for X . If $p(x, n) = g(x, n) \cap h(x, n)$, then $\{p(x, n): x \in X, n \in N\}$ is a graded weak base for X .

PROOF. If F is a closed subset of X and $x \in X - F$, then there exists n such that $h(x, n) \cap F = \emptyset$, whence $p(x, n) \cap F = \emptyset$. Therefore, suppose that B is a subset of X such that if $x \in X - B$, then there exists $n = n(x)$ for which $p(x, n) \cap B = \emptyset$. We must show that B is closed. Assume that B is not closed. Then there exists $x \in \text{cl}(B) - B$ such that $h(x, n) \cap B \neq \emptyset$ for all $n \in N$. Given $g(x, i)$, choose $n_i \geq i$ so that $h(x, n_i) \subset g(x, i)$. Let $b_{n_i} \in B$ with $b_{n_i} \in h(x, n_i)$. Then, $b_{n_i} \in h(x, n_i) \cap g(x, i) \subset p(x, i)$. Thus $p(x, i) \cap B$

$\neq \emptyset$ for all $i \in N$, which is a contradiction.

The notion of a weakly first countable space was introduced in [1, p. 129]. Clearly, a space X is weakly first countable if and only if it has a graded weak base. Using Lemma 1, it is easy to establish the following:

REMARK 2. Every weakly first countable quasi Nagata space is a pseudo Nagata space.

Combining Remarks 1 and 2, we see that every weakly first countable wN-space is a pseudo Nagata space. It is not true, however, that every wN-space is a pseudo Nagata space. For example, if X is the topological product of an uncountable family of closed unit intervals, then X , being compact, is a wN-space; but X is not weakly first countable, hence not a pseudo Nagata space.

Here is an example of a weakly first countable quasi Nagata space which is not a wN-space.

EXAMPLE 1. Let R be obtained from $[0, \infty) = \{x: 0 \leq x < \infty\}$ by identifying $1/2$ and 2 , $1/3$ and 3 , and generally, $1/n$ and n for all $n \in N$. We denote by x_n the point $[1/n, n]$ in the identification space R . All other points of R are singleton equivalence classes, i.e., real numbers. The space R was given in [1, Example 2.2] as an example of a symmetrizable space which is not semi-metrizable.

Throughout the discussion of the space R , we let $S = \{x_n: n \in N\}$, where, as above, x_n denotes the point of R which is the equivalence class $[1/n, n]$. Also, we let $Y = R - S$.

We shall now show that the space R is not a wN-space. Let $\{g(x, n): x \in R, n \in N\}$ be any graded system of open covers for R . Choose an increasing sequence $\{n_i\}$ in N such that $x_{n_i} \in g(0, i)$ for all $i \in N$. Choose $y_i \in Y$ such that $|y_i - n_i| < 1/i$ for all $i \in N$ and such that $y_i \in g(0, i)$. Then $g(y_i, i) \cap g(0, i) \neq \emptyset$ for all $i \in N$, but $\{y_i\}$ does not have a cluster point. It follows that R cannot be a wN-space.

We shall now show that R is a quasi Nagata space. If $y \in Y - \{0\}$, let $\{g(y, n): n \in N\}$ be a local open base at y such that $g(y, n) \cap S = \emptyset$, $g(y, n+1) \subset g(y, n)$, and such that if $x \in g(y, n)$, then $|x - y| < 1/n$. If $x \in S$, let $\{g(x, n): n \in N\}$ be a local open base at x such that $0 \notin g(x, n)$ and $g(x, n+1) \subset g(x, n)$. Let

$$g(0, n) = ([0, 1/n) \cap Y) \cup \bigcup \{H(x_i, n): i = n, n+1, \dots\}$$

where $H(x_i, n) = (x_i - 1/n, x_i) \cup (x_i, x_i + 1/n)$. Then $\{g(x, n): x \in R, n \in N\}$ is a graded system of open covers for R .

Let $y_n \in g(z_n, n)$ for all $n \in N$ and suppose that $\{y_n\}$ converges to a point y . If $z_n \in S$ for infinitely many $n \in N$, then 0 is a cluster point of $\{z_n\}$. Therefore, assume that only finitely many points of $\{z_n\}$ belong to S ; without loss of generality, we may as well assume that $z_n \notin S$ for all $n \in N$. Then $y_n \in Y - \{0\}$ for all $n \in N$. It follows that $\{z_n\}$ converges to y , completing the proof that R is a quasi Nagata space. Since R is weakly first countable, by Remark 2, the space R is also a pseudo Nagata space.

We now establish the following general criterion for metrizable:

THEOREM 1. A Hausdorff γ -space is metrizable if and only if it is either a pseudo Nagata space or a quasi Nagata space.

PROOF. Let X be a Hausdorff pseudo Nagata space (quasi Nagata space) by virtue of a graded weak base (graded system of open covers) $\{g(x, n): x \in X, n \in N\}$, that is, $\{g(x, n)\}$ is a graded weak base (graded system of open covers) for X such that if $\{x_n\}$ is a convergent sequence with $x_n \in g(y_n, n)$ for all $n \in N$, then $\{y_n\}$ has a cluster point. Let X be a γ -space by virtue of a graded open basis $\{h(x, n): x \in X, n \in N\}$, that is, if A is compact and B is closed and disjoint from A , then $h(A, n) \cap B = \emptyset$ for some $n \in N$. Let $p(x, n) = g(x, n) \cap h(x, n)$ for all x in X and all $n \in N$. If $\{g(x, n)\}$ is a graded system of open covers for X , then $\{p(x, n)\}$ is a graded open base for X . If $\{g(x, n)\}$ is a graded weak base for X , then by Lemma 1, $\{p(x, n)\}$ is also a graded weak base for X . In either case, $\{p(x, n)\}$ is a graded weak base for the space X . Let A be compact and B be closed and disjoint from A . Since $\{h(x, n)\}$ is a γ -basis for X , there exists a natural number n such that $h(A, n) \cap B = \emptyset$, so that $p(A, n) \cap B = \emptyset$.

Suppose that $A \cap p(B, n) \neq \emptyset$ for all $n \in N$. Then there exist sequences $\{a_n\}$ in A and $\{b_n\}$ in B with $a_n \in p(b_n, n)$ for all $n \in N$. Since X is first countable and A is compact, the sequence $\{a_n\}$ has a convergent subsequence, say $\{a_{n_i}\}$. We have $a_{n_i} \in p(b_{n_i}, n_i)$ for all $i \in N$. Let $d_i = b_{n_i}$ and $c_i = a_{n_i}$ for $1 \leq i \leq n_1$; let $d_i = b_{n_2}$ and $c_i = a_{n_2}$ for $n_1 < i \leq n_2$; generally, let $d_i = b_{n_j}$ and $c_i = a_{n_j}$ for $n_{j-1} < i \leq n_j$ for all $j > 1$. We have $\{c_n\}$ being a convergent sequence and $c_n \in p(d_n, n)$ for all $n \in N$. It follows that $\{d_n\}$ has a cluster point so that $\{d_n\}$ has a subsequence $\{d_{n_j}\}$ such that $\{d_{n_j}\}$ converges to a point b in B . Let $S = \{b\} \cup \{d_{n_j}: j \in N\}$. Since X is Hausdorff, the set A is closed. Since S is compact, there exists an integer k for which

$$p(S, k) \cap A = \emptyset.$$

Then $c_{n_j} \notin p(d_{n_j}, k)$ for all $j \in N$. Choose $n_j > k$; then $p(d_{n_j}, n_j) \subset p(d_{n_j}, k)$ and we have the contradiction $c_{n_j} \notin p(d_{n_j}, n_j)$. It follows that $A \cap p(B, n) = \emptyset$ for some n .

Define a symmetric d for X as follows: if x and y are distinct points of X , let $d(x, y) = 1/n$ where n is the least natural number k such that x is not in $p(y, k)$ and y is not in $p(x, k)$. A subset of X is said to be d -closed provided that whenever $x \in X - K$, we have $d(x, K) > 0$. The complements of the d -closed sets form a topology T_d for the set X . From the fact that $\{p(x, n)\}$ is a weak base for X and that if F is closed, then $F = \bigcap \{p(F, n): n \in N\}$, it follows easily that T_d coincides with the original topology on X , that is, X is a symmetrizable space and d is an admissible symmetric for X .

If A is compact and B is closed and disjoint from A , then there exists an integer n such that $p(A, n) \cap B = \emptyset$ and $p(B, n) \cap A = \emptyset$. It follows that $d(A, B) > 0$. The metrizability of the space X now follows by Theorem 3.1 of [1] or Theorem 2 of [10], completing the proof.

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