FOURIER TRANSFORMS OF B -SPLINES AND FUNDAMENTAL SPLINES FOR CARDINAL HERMITE INTERPOLATIONS

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ABSTRACT. Using the exponential Hermite Euler splines we compute the Fourier transforms of the B -splines and fundamental splines for Cardinal Hermite Interpolation, introduced by Schoenberg and Sharma and Lipow and Schoenberg respectively.

Introduction. Let n, r be positive integers such that $n > 2r - 1$. The class $\mathcal{S}_{n,r}$ of cardinal splines of degree n with integer knots of multiplicity r consists of functions $S(x)$ which are polynomials of degree n in each of the intervals $[\nu, \nu + 1]$ ($\nu = 0, \pm 1, \pm 2, \dots$) and belong to class $C^{n-r}(-\infty, \infty)$. The Cardinal Hermite Interpolation Problem (C.H.I.P.) was first considered by P. Lipow and I. J. Schoenberg [1] whose main result states that for any set of r bi-infinite sequences $(y_\nu^{(s)})$ ($s = 0, 1, \dots, r - 1$) such that

$$(1) \quad y_\nu^{(s)} = O(|\nu|^\gamma) \quad (s = 0, 1, \dots, r - 1) \text{ for some } \gamma > 0,$$

there is a unique $S_{2m-1}(x) \in \mathcal{S}_{2m-1,r}$ such that

$$(2) \quad S_{2m-1}^{(s)}(\nu) = y_\nu^{(s)} \quad (s = 0, 1, \dots, r - 1) \text{ for all integers } \nu.$$

Furthermore, the spline $S_{2m-1}(x)$ is given by the Hermite interpolation formula

$$(3) \quad S_{2m-1}(x) = \sum_{s=0}^{r-1} \sum_{\nu=-\infty}^{\infty} y_\nu^{(s)} L_{2m-1,r,s}(x - \nu),$$

where the fundamental splines $L_{2m-1,r,s}(x)$ ($s = 0, 1, \dots, r - 1$) are uniquely determined by the conditions

$$(4) \quad \begin{cases} L_{2m-1,r,s}^{(\rho)}(\nu) = 0 & (\rho = 0, 1, \dots, r - 1) \quad \forall \text{ integers } \nu \neq 0 \\ L_{2m-1,r,s}^{(\rho)}(0) = \delta(\rho, s), & \text{Kronecker delta.} \end{cases}$$

Subsequently, Schoenberg and Sharma [11] introduced the B -splines $N_s(x)$ ($s = 0, 1, \dots, r - 1$) which belong to the space $\mathcal{S}_{2m-1,r}^{(s)} = \{S(x) \in \mathcal{S}_{2m-1,r} : S^{(\rho)}(\nu) = 0 \text{ } (\rho = 0, 1, \dots, r - 1, \rho \neq s) \text{ } \forall \text{ integers } \nu\}$, have support in $-(m - r + 1), (m - r + 1)$ and satisfy the interpolatory properties

$$(5) \quad N_s^{(s)}(\nu) = \begin{cases} C_\nu & (\nu = -(m - r), \dots, (m - r)) \\ 0 & \text{otherwise,} \end{cases}$$

where C_ν are the coefficients of the monic reciprocal polynomials

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$$(6) \quad \Pi_{2m-1,r}(\lambda) = \sum_{\nu=0}^{2m-2r} C_{\nu-(m-r)} \lambda^{\nu} \quad (C_0 > 0).$$

One of the most important properties of the B -splines (see [2], [11]) asserts that every $S(x) \in \mathcal{S}_{2m-1,r}^{(s)}$ admits a unique representation of the form

$$(7) \quad S(x) = \sum_{\nu=-\infty}^{\infty} a_{\nu} N_s(x - \nu).$$

For the case $r = 1$, the corresponding interpolation problem, called the Cardinal Interpolation Problem (C.I.P.) dates back to the first paper on spline functions of I. J. Schoenberg [6] and has been extensively studied by him in several recent works (see [7], [8], [10]). The theory of C.I.P. (the case $r = 1$) is very much enriched by the help of the Fourier transforms of the B -splines and fundamental splines (see [5], [6], [7], [12], [13]) viz.

$$(8) \quad N_0(x) = \frac{(2m-1)!}{2\pi} \int_{-\infty}^{\infty} e^{iux} \left\{ \frac{2 \sin u/2}{u} \right\}^{2m} du,$$

$$(9) \quad L_{2m-1,1,0}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \frac{u^{-2m}}{\alpha_{2m}(u)} du,$$

where

$$(10) \quad \alpha_n(u) = \sum_{k=-\infty}^{\infty} \frac{1}{(u + 2\pi k)^n}.$$

The purpose of this paper is to compute Fourier transforms of the fundamental splines $L_{2m-1,r,s}(x)$ and B -splines $N_s(x)$ for the general C.H.I.P. This is achieved with the help of the exponential Hermite Euler splines introduced in [3]. More precisely, we shall prove the following theorems.

THEOREM 1. *The Fourier integral representations of the fundamental splines are given by*

$$(11) \quad L_{2m-1,r,s}(x) = \frac{(-i)^s}{2\pi} \int_{-\infty}^{\infty} e^{iux} \frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))} du \quad (s = 0, 1, \dots, r-1)$$

where $H_r(a_n)$ denotes the Hankel determinant given by

$$(12) \quad H_r(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r+1} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-r+1} & a_{n-r} & \cdots & a_{n-2r+2} \end{vmatrix}$$

and $H_{r,s}(\alpha_{2m}(u))$ is obtained from $H_r(\alpha_{2m}(u))$ by replacing the $(s+1)$ th column ($s = 0, 1, \dots, r-1$) by the column vector

$$(1/u^{2m}, 1/u^{2m-1}, \dots, 1/u^{2m-r+1})^T.$$

THEOREM 2. *The Fourier integral representations of the B -splines are given by*

$$(13) \quad N_s(x) = (-i)^s \frac{K(m, r)}{2\pi} \int_{-\infty}^{\infty} e^{iux} \left(2 \sin \frac{u}{2}\right)^{2m} H_{r,s}(\alpha_{2m}(u)) du$$

where

$$(14) \quad K(m, r) = (-1)^{m(r+1)} \frac{(2m-1)! (2m-2)! \cdots (2m-r)!}{1! 2! \cdots (r-1)!}$$

In §1 we give the preliminaries required to prove Theorems 1 and 2, while in the last section we mention a formula to approximate the Fourier transform of a given function.

1. Preliminaries. The tool for our computations of the Fourier transforms (11) and (13) are the exponential Hermite Euler splines $S_{n,r,s}(x; \lambda)$ ($s = 0, 1, \dots, r-1$) defined by

$$(1.1) \quad \begin{cases} S_{n,r,s}(x; \lambda) = A_{n,r,s}(x; \lambda) / H_r(A_n(0; \lambda) / n!) & (0 \leq x \leq 1) \text{ and} \\ S_{n,r,s}(x+1; \lambda) = \lambda S_{n,r,s}(x; \lambda) & \forall \text{ real } x, \end{cases}$$

where

$$A_{n,r,s}(x; \lambda) = \begin{bmatrix} \frac{A_n(0; \lambda)}{n!} & \cdots & \frac{A_{n-s+1}(0; \lambda)}{(n-s+1)!} & \frac{A_n(x; \lambda)}{n!} & \frac{A_{n-s-1}(0; \lambda)}{(n-s-1)!} & \cdots & \frac{A_{n-r+1}(0; \lambda)}{(n-r+1)!} \\ \frac{A_{n-1}(0; \lambda)}{(n-1)!} & \cdots & \frac{A_{n-s}(0; \lambda)}{(n-s)!} & \frac{A_{n-1}(x; \lambda)}{(n-1)!} & \frac{A_{n-s-2}(0; \lambda)}{(n-s-2)!} & \cdots & \frac{A_{n-r}(0; \lambda)}{(n-r)!} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{A_{n-r+1}(0; \lambda)}{(n-r+1)!} & \cdots & \frac{A_{n-r-s+2}(0; \lambda)}{(n-r-s+2)!} & \frac{A_{n-r+1}(x; \lambda)}{(n-r+1)!} & \frac{A_{n-r-s}(0; \lambda)}{(n-r-s)!} & \cdots & \frac{A_{n-2r+2}(0; \lambda)}{(n-2r+2)!} \end{bmatrix}$$

and $A_n(x; \lambda)$ are the exponential Euler polynomials (see [9]). Furthermore $S_{n,r,s}(x; \lambda)$ satisfy the following interpolatory conditions (see [3]):

$$(1.2) \quad \begin{cases} S_{n,r,s}^{(\rho)}(\nu; \lambda) = 0 & (\rho = 0, 1, \dots, r-1, \rho \neq s) \text{ and} \\ S_{n,r,s}^{(s)}(\nu; \lambda) = \lambda^\nu & (\nu = 0, \pm 1, \pm 2, \dots). \end{cases}$$

We shall also need the following relation given in [4]:

$$(1.3) \quad H_r(A_n(0; \lambda) / n!) = (-1)^{\lfloor r/2 \rfloor + (r-1)(n+1)} \frac{C(n, r) \Pi_{n,r}(\lambda)}{(\lambda - 1)^{n-r+1}},$$

where $C(n, r) = 1! 2! \cdots (r-1)! / n!(n-1)! \cdots (n-r+1)!$, and $\Pi_{n,r}(\lambda)$ are given by (6).

2. Proof of Theorem 1. Consider the bounded exponential Hermite Euler splines $S_{2m-1,r,s}(x; e^{iu})$ for $0 < u < 2\pi$ if r is even and $-\pi < u < \pi$ if r is odd. Following Schoenberg it is easy to show that the functions

$$(2.1) \quad \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} S_{2m-1,r,s}(x; e^{iu}) du & (r \text{ even}) \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{2m-1,r,s}(x; e^{iu}) du & (r \text{ odd}) \end{cases}$$

are spline functions belonging to $\mathbb{S}_{2m-1,r}$. Furthermore, these functions satisfy (4) in view of the interpolating conditions $S_{2m-1,r,s}^{(s)}(\nu; e^{iu}) = e^{i\nu u}$ and $S_{2m-1,r,s}^{(\rho)}(\nu; e^{iu}) = 0$ ($\rho = 0, 1, \dots, r-1, \rho \neq s$) for all integers ν . Hence, for $s = 0, 1, \dots, r-1$,

$$(2.2) \quad L_{2m-1,r,s}(x) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} S_{2m-1,r,s}(x; e^{iu}) du & (r \text{ even}) \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{2m-1,r,s}(x; e^{iu}) du & (r \text{ odd}). \end{cases}$$

Using the expansion

$$(2.3) \quad \frac{A_n(x; e^{iu})}{n!} = (e^{iu} - 1)e^{-iux} \sum_{k=-\infty}^{\infty} \frac{e^{2\pi i k x}}{(ui + 2\pi ki)^{n+1}} \quad (0 \leq x \leq 1)$$

(see [9]), it follows from (1.1) that

$$(2.4) \quad S_{2m-1,r,s}(x; e^{iu})e^{-iux} = (-i)^s \sum_{k=-\infty}^{\infty} e^{2\pi i k x} \frac{\Delta_{2m,k,s}(x; u)}{H_r(\alpha_{2m}(u))},$$

where $\Delta_{2m,k,s}(x; u)$ is obtained from $H_r(\alpha_{2m}(u))$ by replacing the s th column by the column vector

$$\left(\frac{1}{(u + 2\pi k)^{2m}}, \frac{1}{(u + 2\pi k)^{2m-1}}, \dots, \frac{1}{(u + 2\pi k)^{2m-r+1}} \right)^T.$$

Substituting (2.3) into (2.2) and performing a change of variable after interchanging the order of integration and summation we obtain (11).

REMARK. When $r = 1$ (in which case s can take only the value zero), $H_{1,0}(\alpha_{2m}(u)) = u^{-2m}$ and so (11) reduces to (9).

3. Proof of Theorem 2. From (3) and (5) we can write

$$N_s(x) = \sum_{\nu=-(m-r)}^{(m-r)} C_{\nu} L_{2m-1,r,s}(x - \nu),$$

and using (11), after some simplifications, we obtain

$$(3.1) \quad N_s(x) = \frac{(-i)^s}{2\pi} \int_{-\infty}^{\infty} e^{-(m-r)iu} \Pi_{2m-1,r}(e^{iu}) e^{iux} \frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))} du.$$

Using (1.3), we can write

$$\Pi_{2m-1,r}(e^{iu}) = (-1)^{\lfloor r/2 \rfloor + m(r+1)} K(m, r) (e^{iu} - 1)^{2m-r} H_r(A_n(0; e^{iu})/n!).$$

Then using expansion (2.3) we obtain

$$(3.2) \quad \Pi_{2m-1,r}(e^{iu}) = (-i)^{2mr} K(m, r) \frac{(e^{iu} - 1)^{2m}}{e^{iur}} H_r(\alpha_{2m}(u)),$$

where $K(m, r)$ is given by (14). Substituting (3.2) into (3.1), we obtain (13).

REMARK. When $r = 1$ (in which case $s = 0$), (13) reduces to (8).

4. An approximation of Fourier transform. Let $f(x)$ be $(2m - r + 1)$ times continuously differentiable and $f^{(s)}(x) \in L_1(\mathbf{R})$ ($s = 0, 1, \dots, r - 1$). Using the same method as Silliman [13] it is possible to obtain a formula to approximate the Fourier transform

$$(4.1) \quad F(u) = \int_{-\infty}^{\infty} e^{iux} f(x) dx.$$

Indeed, let $S_{2m-1}(x)$ be the unique solution of the C.H.I.P. to the data $\{f^{(s)}(\nu)\}$ ($s = 0, 1, \dots, r - 1$). Then

$$(4.2) \quad S_{2m-1}(x) = \sum_{\nu=-\infty}^{\infty} \sum_{s=0}^{r-1} f^{(s)}(\nu) L_{2m-1,r,s}(x - \nu).$$

Using Theorem 1, the Fourier transform of $S_{2m-1}(x)$ is given by

$$(4.3) \quad \int_{-\infty}^{\infty} S_{2m-1}(x) e^{iux} dx = \sum_{s=0}^{r-1} (-1)^s \frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))} \sum_{\nu=-\infty}^{\infty} f^{(s)}(\nu) e^{i\nu u},$$

which serves as an approximation to $F(u)$. We can write

$$(4.4) \quad \int_{-\infty}^{\infty} e^{iux} f(x) dx = \sum_{s=0}^{r-1} (-1)^s \frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))} \sum_{\nu=-\infty}^{\infty} f^{(s)}(\nu) e^{i\nu u} + R_f(u)$$

where $R_f(u)$ is the error term.

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